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Problems in viscoelasticity

Albert William Zechmann
Iowa State University

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Albert William Zechmann

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Head of Major Department

Signature was redacted for privacy.

Dean of Graduate College

Iowa State University
Of Science and Technology
Ames, Iowa

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NOTATION

$c_{ijk\ell}$, $C_{ijk\ell}$ = constants which characterize the elastic properties of a material. The first gives stresses in terms of strains; the second gives strains in terms of stresses.

$d_{ijk\ell}$, $D_{ijk\ell}$ = constants which characterize the viscous properties of a material.

e_{ij} = components of the deviatoric strain tensor.

E = Young's Modulus, spring constant.

F_i, F_i' = body force components per unit mass.

I_1, I_2, I_3 = scalar invariants of the strain.

J = compliance.

L = Laplace transform operator.

L^{-1} = inverse Laplace transform operator.

Q, Q', P, P' = viscoelastic operators.

S_{ij} = components of the deviatoric stress tensor.

T_i, T_i', T_i'' = surface traction components.

u_i, v_i, w_i = components of displacement vectors.

α_{ij} = second order components of pure strain.

γ^*_{ij} = components of the total strain tensor,

γ_{ij} = components of the total pure strain tensor.

$\delta(t)$ = delta function.

δ_{ij} = Kronecker delta.

ϵ = one-dimensional strain.

ϵ_{ij} = components of the linear strain tensor.

η = constant coefficient of viscosity, dashpot constant.

μ, λ = Lame constants.

ρ_0, ρ = density.

σ_{ij}, σ'_{ij}, σ''_{ij} = components of stress tensors.

τ = one-dimensional stress.

ξ'_i = relaxation time.

ξ_i = retardation time.

τ_{ij}, τ''_{ij} = components of linear stress tensors.

τ'_{ij} = components of the second order stress tensor.

ϕ, $\phi_{ijk\ell}$ = recollection functions for relaxation.

ψ, $\psi_{ijk\ell}$ = memory functions for creep.

n_i = components of the unit normal vector.

INTRODUCTION

Viscoelasticity is one branch of rheology concerned with time-dependent mechanical behavior; it studies the behavior of media possessing both elastic and viscous properties. The behavior of such diverse materials as cement, high-polymers, wet sponges, toothpaste, and mud are included in its domain.

In the past, attention has been focused primarily on the phenomenological aspects of time-dependent elasticity. Various mechanical models consisting of intricate arrangements of numerous springs and dashpots have been constructed which respond almost as the materials themselves under analogous situations. One of the best works illustrating this approach is due to Alfrey [1][1]. Most of the phenomenological development, however, has been directed toward the formulation of one-dimensional time-dependent stress-strain laws. Just recently Bland [2] succeeded in extending this treatment to three dimensions.

Functions such as the creep function, relaxation function, distribution function for relaxation times, and the distribution function for retardation times may be selected to facilitate the description of time-dependent elastic behavior. Gross [3] has shown the connection between the various de-

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[1]Numbers in square brackets refer to the bibliography at the end of the thesis.

scriptive functions from a theoretical point of view.

Less work has been done in the application of these stress-strain laws to particular problems. One method for solving quasi-static problems involving only infinitesmal deformations, the Laplace transform method, is due to Lee [4] ; another method, the method of functional equations, formulated by Radok [5], is intended for those problems not amenable to transform methods.

Yet, whenever viscous flow occurs, few problems can be classified as ones involving infinitesmal deformations. The compression of a wet sponge is obviously out of the domain of infinitesmal theory. A more powerful method of solution is needed in order to solve such problems. Murnaghan [6], Green and Zerna [7] , and Rivlin [8] have furnished several methods of solving problems in classical finite elasticity. Some work has also been done on finite deformations of time dependent media, but on a highly theoretical level.

The present paper attempts to extend the methods of second order elastic theory to similar problems involving viscoelastic materials. In order to do this, it is necessary to redevelop the techniques employed in the solution of linear problems. After formulating an approximate method of solution of the viscoelastic problem, several examples are studied in order to determine 1) the effect on the solution of neglecting the inertia effects; 2) the dependence of the method of solution on the type of material; and 3) the time interval over which the solution is valid.

THE PHENOMENOLOGICAL APPROACH

The phenomenon of one-dimensional viscoelasticity can best be described by the results obtained from two different types of experiments. One of these relates the strain to a constant stress, the other, the stress to a constant strain. As a result of numerous experiments of these types described in the literature, the behavior of many viscoelastic materials can be briefly described as below.

A. Elongation under a uniaxial tensile stress:

1. Under the action of a constant uniaxial stress a bar of constant cross-section may

a. first extend elastically, then continue to elongate with time; or

b. initially refuse to elongate, but then do so as a function of time. (In either case the time dependent part of the elongation may approach an asymptotic value.)

2. If the constant load is suddenly removed, a bar can

a. retain the entire deformation,

b. lose some or all of the previous elongation.

B. Stress behavior under a prescribed elongation:

1. Under the action of a constant elongation applied to a bar, the stress in the bar may

a. relax asymptotically to some final stress state;

b. seemingly become infinite (the bar refuses to elongate instantaneously).

2. If the bar is now quickly compressed to its original length, then the stress in the bar tends toward zero, usually.

Finally one could apply a known rate of strain to a bar, and observe the stress response both as a function of strain rate and of strain (strain being analogous to time). A similar procedure could be followed with the rate of stress, but neither experiment will be analysed any further.

If the discussion is limited to the one-dimensional response of linear, homogeneous, isotropic media, then certain names are usually associated with the various viscoelastic phenomena.

A Maxwell material is any material whose one-dimensional viscoelastic behavior is depicted by a stress-strain relationship of the type

$$\dot{\epsilon} = \frac{1}{E}\dot{\tau} + \frac{1}{\eta}\tau \quad , \tag{1}$$

where the dot indicates differentiation with respect to time, e. g.,

$$\dot{\epsilon} = \frac{\partial \epsilon}{\partial t} .$$

Under the action of a constant uniaxial tension such a material first behaves elastically, then flows viscously; that is, the material seems to creep. On the other hand, under the action of a constant strain the stress relaxes; in fact, the stress approaches zero asymptotically (the phenomenon of relaxation). The behavior of concrete is approximately that of a Maxwell material.

A Kelvin-Voigt material is one having a stress-strain relationship of the type

$$\tau = E\epsilon + \eta\dot{\epsilon} \,. \tag{2}$$

Such a material will show no instantaneous deformation corresponding to an applied stress, but approaches the elastic deformation asymptotically in time; the material has the property of an elastic after effect (delayed elasticity). Furthermore a Kelvin material cannot be deformed initially by a finite stress, hence, to obtain additional experimental results the stress-strain-rate behavior must be investigated. An example of a Kelvin-Voigt material is a water saturated sponge. A sponge can be compressed until the water has been driven out (but not instantaneously); at this point the sponge becomes elastic and nearly incompressible.

A Bingham solid is a material which has a stress-strain relation of the form

$$\begin{aligned} &\epsilon = 0, \quad \tau < \zeta \,; \\ &\tau = \zeta + \eta\dot{\epsilon}, \quad \tau \geq \zeta \,; \end{aligned} \tag{3}$$

Expressed in words, no deformation is possible until the stress attains a certain specified value ζ. When the stress exceeds ζ, viscous flow occurs. A good example of a Bingham material is toothpaste.

To each of the above media one can associate a mechanical model, that is, a model which satisfies the same differential equation. The mechanical model consists of a system of springs and dashpots in which the spring constant E is identified with Young's modulus, and η, the viscosity of the dashpot, is identified with the coefficient of viscosity. Normally one is concerned with force-extension relationships in a mechanical system, but this offers no problem; for in the one-dimensional case the bar is assumed to have a constant cross-sectional area which allows the formulation of the following correspondence:

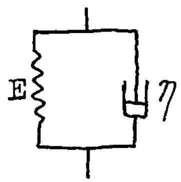
$$\tau = \frac{F}{A}, \qquad \epsilon = \frac{a}{\ell}, \tag{4}$$

where F = force,
A = cross-sectional area,
a = extension,
ℓ = length.

Table 1 lists the three mechanical models corresponding to the media described above, and gives their response to constant load, constant elongation, constant strain rate, and constant stress rate. The methods by which the differential equations are solved and the initial conditions introduced into the solution are discussed in detail later in connection with the four-parameter model.

The three properties which in general characterize a viscoelastic material are

Table 1. Comparison of three mechanical models

Material and corresponding mechanical model	Stress-strain law	Strain response to a constant stress τ_0	Stress response to a constant strain ϵ_0	Strain response to a constant stress rate $\dot{\tau}_0$	Stress response to a constant strain rate $\dot{\epsilon}_0$
 Maxwell	$\dot{\epsilon} = \frac{1}{E}\dot{\tau} + \frac{1}{\eta}\tau$	$\epsilon = \tau_0(\frac{1}{E} + \frac{1}{\eta}t)$	$\tau = \epsilon_0 E e^{-\frac{E}{\eta}t}$	$\epsilon = \frac{\dot{\tau}_0}{2\eta}t^2 + (\frac{1}{E}\dot{\tau}_0 + \frac{\alpha}{\eta})t+\beta$ α, β constants	$\tau = \dot{\epsilon}_0 (1-e^{-\frac{E}{\eta}t})$
 Kelvin-Voigt	$\tau = E\epsilon + \eta\dot{\epsilon}$	$\epsilon = \frac{\tau_0}{E}(1-e^{-\frac{E}{\eta}t})$	$\tau = \epsilon_0[E + \eta\delta(t)]$	$\epsilon = \frac{\dot{\tau}_0}{E^2}e^{-\frac{E}{\eta}t} + \frac{E}{\eta}(t - \frac{\eta}{E})$	$\tau = \dot{\epsilon}_0(Et+\eta)$
 Frictional Restraint Bingham Solid	$\epsilon = 0,\ \tau_0 < \zeta$ $\tau = \zeta + \eta\dot{\epsilon},$ $\tau_0 \geq \zeta$	$\epsilon = 0,\ \tau_0 < \zeta$ $= \frac{\tau_0 - \zeta}{\eta}t,$ $\tau_0 \geq \zeta$	$\tau = \epsilon_0[\eta\delta(t) + \zeta]$	$\epsilon = 0,\ \tau_0 = \dot{\tau}_0 t < \zeta$ $\epsilon = \frac{1}{\eta}(\frac{\dot{\tau}_0 t}{2} - \zeta)t,$ $\tau_0 = \dot{\tau}_0 t \geq \zeta$	$\tau = \zeta + \eta\dot{\epsilon}_0$

1) instantaneous elasticity,

2) retarded elasticity,

3) viscous flow.

These three attributes can be combined in either of the four element mechanical models shown in Figure 1.

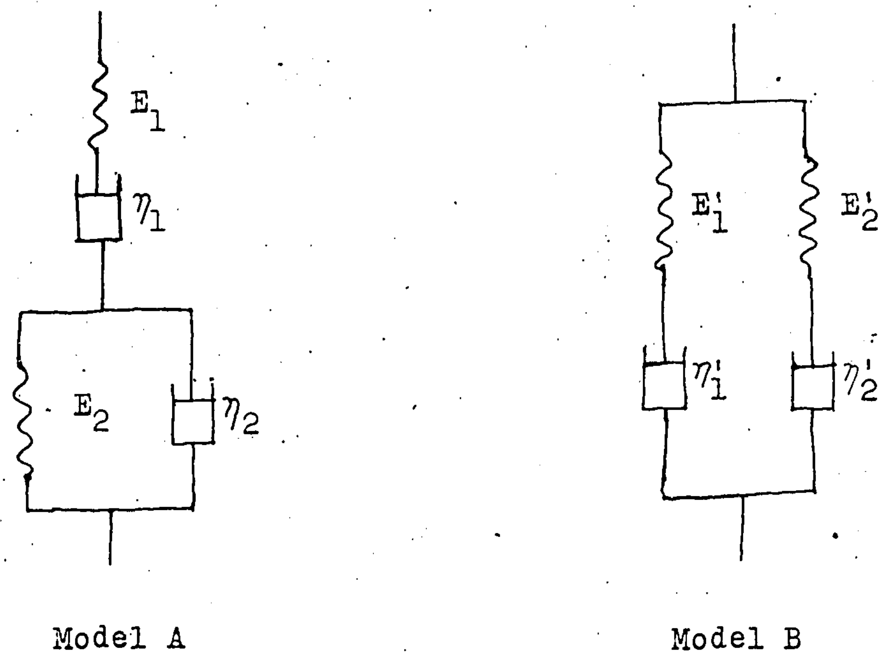


Figure 1. Four-parameter models

The relationship between model A and the three visco-elastic properties listed above is quite apparent when the spring E_1 is associated with instantaneous elasticity, the Voigt element (η_2, E_2) with retarded elasticity, and the dashpot η_1 with viscous flow. Model B in turn is equivalent to model A, but cannot be readily interpreted in terms of the viscoelastic properties.

Model A consists of a Maxwell element in series with a Voigt element. The stress is the same in both elements, and the strain is the sum of the strains, $\epsilon_1 + \epsilon_2$, of the individual elements. For the Maxwell element

$$(D + \frac{E_1}{\eta_1})\tau = E_1 D \epsilon_1, \tag{5}$$

where

$$D = \frac{\partial}{\partial t};$$

and for the Voigt element

$$\frac{1}{\eta_2} = (D + \frac{E_2}{\eta_2}) \epsilon_2. \tag{6}$$

Multiply (5) by $(D + \frac{E_2}{\eta_2})$, (6) by $(D + \frac{E_1}{\eta_1})$, and add. Then

$$D^2\tau + (\frac{E_1}{\eta_1} + \frac{E_1}{\eta_2} + \frac{E_2}{\eta_2})D\tau + \frac{E_1 E_2}{\eta_1 \eta_2}\tau = E_1 D^2 \epsilon + \frac{E_1 E_2}{\eta_2} D\epsilon . \tag{7}$$

Model B consists of two Maxwell elements in parallel. In this model the strain is identical in each element, and the total stress is the sum $(\tau_1 + \tau_2)$ of the individual stresses. The stress-strain relationship is given by the system

$$\begin{aligned} (D + \frac{E_1'}{\eta_1'})\tau_1 &= E_1' D\epsilon , \\ (D + \frac{E_2'}{\eta_2'})\tau_2 &= E_2' D\epsilon . \end{aligned} \tag{8}$$

Combine the two equations (8):

$$(9)\quad (D + \frac{E_1'}{\eta_1'})(D + \frac{E_2'}{\eta_2'})\tau = \left[E_1'(D + \frac{E_2'}{\eta_2'}) + E_2'(D + \frac{E_1'}{\eta_1'})\right] D\epsilon \, ;$$

or if (9) is expanded,

$$(10)\qquad D^2\tau + (\frac{E_1'}{\eta_1'} + \frac{E_2'}{\eta_2'})D\tau + \frac{E_1'E_2'}{\eta_1'\eta_2'}\tau =$$

$$(E_1' + E_2')D^2\epsilon + E_1'E_2'(\frac{1}{\eta_1'} + \frac{1}{\eta_2'})D\epsilon \, .$$

Since the differential equations (7) and (10) are of the same form, E_1, E_2, η_1, η_2 can be expressed in terms of E_1', E_2', η_1', η_2' and vice-versa.

$$(11)\qquad \begin{aligned} E_1 &= E_1' + E_2', \\ E_2 &= \frac{E_1'E_2'(\eta_1' + \eta_2')^2(E_1' + E_2')}{(\eta_1'E_2' - \eta_2'E_1')^2}, \\ \eta_1 &= \eta_1' + \eta_2', \\ \eta_2 &= \frac{\eta_1'\eta_2'(E_1' + E_2')^2(\eta_1' + \eta_2')}{(\eta_1'E_2' - \eta_2'E_1')^2}. \end{aligned}$$

$$(12)\qquad \begin{aligned} E_1' &= \frac{1}{2\beta}\left[E_1(\alpha + \beta) - E_1^2(\frac{1}{\eta_1} + \frac{1}{\eta_2})\right], \\ E_2' &= -\frac{1}{2\beta}\left[E_1(\alpha - \beta) - E_1^2(\frac{1}{\eta_1} + \frac{1}{\eta_2})\right], \\ \eta_1' &= \frac{1}{2\beta}\left[E_1(\alpha + \beta) - E_1^2(\frac{1}{\eta_1} + \frac{1}{\eta_2})\right]\frac{1}{(\alpha - \beta)}, \end{aligned}$$

$$\eta_2' = -\frac{1}{2\beta}\left[E_1(\alpha - \beta) - E_1^2\left(\frac{1}{\eta_1} + \frac{1}{\eta_2}\right)\right]\frac{1}{(\alpha + \beta)},$$

where

$$\alpha = \frac{1}{2}\left(\frac{E_1}{\eta_1} + \frac{E_1}{\eta_2} + \frac{E_2}{\eta_2}\right),$$

$$\beta^2 = \alpha^2 - \frac{E_1E_2}{\eta_1\eta_2}.$$

By considering various arrangements of two dashpots and two springs, other models can be formed, yet all but one are either equivalent to model A, or to a simpler model. The one new model is a double Voigt model which is a combination of two Voigt elements in series. In this latter case,

$$\frac{1}{\eta_1}\tau = \left(D + \frac{E_1}{\eta_1}\right)\epsilon_1,$$

(13)

$$\frac{1}{\eta_2}\tau = \left(D + \frac{E_2}{\eta_2}\right)\epsilon_2,$$

hence

(14) $$D^2\epsilon + \left(\frac{E_1}{\eta_1} + \frac{E_2}{\eta_2}\right)D\epsilon + \frac{E_1E_2}{1\ 2}\epsilon =$$

$$\left(\frac{1}{\eta_1} + \frac{1}{\eta_2}\right)D\tau + \frac{1}{\eta_1\eta_2}(E_1 + E_2)\tau,$$

a differential expression of a different form than previously encountered.

The usefulness of two models to describe the same phenomenon can be seen by analysing two viscoelastic problems. The first problem is to determine the strain arising from a prescribed stress, and the second is to determine the stress given a particular strain history. In order to specify the problem completely, a set of initial conditions must also be stated. These initial conditions must, of course, be physically realistic. Thus, if the body is in an undisturbed state, the application of a set of displacements, or surface tractions at t = 0 must produce initial elastic stresses and displacements (provided that any displacement is possible).

When the stress is specified as a continuous function of t, the strain can be determined quite simply from model A using (5), (6) and the initial elastic conditions

$$\tau(0) = \tau_0, \quad \epsilon_1(0) = \frac{\tau_0}{E_1}, \quad \epsilon_2(0) = 0. \tag{15}$$

The solution of this problem can be simplified by the use of the Laplace Transform which is defined by the equation

$$L\left[F(t)\right] = f(s) = \int_0^\infty e^{-st}F(t)dt.$$

Take the Laplace transform of (5) and (6):

$$\overline{\epsilon_1}(s) = \frac{1}{E_1}(s + \frac{E_1}{\eta_1})\frac{\overline{\tau}(s)}{s} + \frac{1}{s}\left[\epsilon_1(0) - \frac{\tau(0)}{E_1}\right],$$

and

$$\overline{\epsilon_2}(s) = \frac{1}{\eta_2}\,\frac{\overline{\tau}(s)}{(s + \frac{E_2}{\eta_2})} + \frac{\epsilon_2(0)}{(s + \frac{E_2}{\eta_2})}.$$

The initial terms drop out when initial elastic conditions are specified thereby reducing the transformed problem to the form

$$(16)\qquad \begin{aligned} \overline{\epsilon_1}(s) &= \frac{1}{E_1}\left(s + \frac{E_1}{\eta_1}\right)\frac{\overline{\tau}(s)}{s}, \\ \overline{\epsilon_2}(s) &= \frac{1}{\eta_2}\,\frac{\overline{\tau}(s)}{\left(s + \frac{E_2}{\eta_2}\right)}. \end{aligned}$$

By the definition of $\epsilon(t)$ and the linearity of the transform,

$$(17)\qquad \overline{\epsilon}(s) = \left[\frac{1}{E_1} + \frac{1}{\eta_1 s} + \frac{1}{\eta_2}\,\frac{1}{\left(s + \frac{E_2}{\eta_2}\right)}\right]\overline{\tau}(s).$$

When the strain is prescribed as a continuous function of t, then model B is more convenient for determining the stress, using (8) and the initial elastic conditions

$$(18)\qquad \epsilon(0) = \epsilon_0, \quad \tau_1(0) = E_1\,\epsilon_0, \quad \tau_2(0) = E_2\,\epsilon_0.$$

The resulting transformed problem,

$$\overline{\tau_1}(s) = \frac{1}{\left(s + \frac{E_1'}{\eta_1'}\right)}\left[E_1' s\,\overline{\epsilon}(s) - E_1'\,\epsilon(0) + \tau_1(0)\right],$$

$$\overline{\tau_2}(s) = \frac{1}{\left(s + \frac{E_2'}{\eta_2'}\right)}\left[E_2' s\,\overline{\epsilon}(s) - E_2'\,\epsilon(0) + \tau_2(0)\right],$$

again simplifies since the initial elastic terms cancel out. Therefore,

$$(19)\qquad \overline{\tau}(s) = \left[E_1'\frac{1}{\left(s + \frac{E_1'}{\eta_1'}\right)} + E_2'\frac{1}{\left(s + \frac{E_2'}{\eta_2'}\right)}\right] s\,\overline{\epsilon}(s).$$

Equations (7) and (10) are also descriptions of the same phenomena; hence, their transforms must be identical with (17) and (19) respectively. This, of course, means that the initial terms must again cancel out, but does not necessarily imply zero initial conditions. Let (7) and (10) be represented simultaneously by the expression

(20) $$D^2\tau + AD\tau + B\tau = CD^2\epsilon + GD\epsilon,$$

where A, B, C and G are all constants. The transform of (20) is

(21) $$(s^2 + As + B)\overline{\tau}(s) = (Cs + G)s\overline{\epsilon}(s),$$

and the initial conditions satisfy

(22) $$(s + A)\tau(0) + \dot{\tau}(0) = (Cs + G)\epsilon(0) + C\dot{\epsilon}(0).$$

Either viscoelastic problem ($\tau(t)$ prescribed or $\epsilon(t)$ prescribed) can be solved using (20), but the initial conditions are already inherent in the model and cannot be specified arbitrarily as a companion to differential equations (7) or (10). Suppose that $\tau(t)$ is prescribed; then since (20) is a linear second order differential equation in $\epsilon(t)$, one would expect $\epsilon(0)$ and $\dot{\epsilon}(0)$ to be prescribed as initial conditions. The physical requirement of elastic initial conditions furnishes $\epsilon(0)$ from $\tau(0)$, while $\dot{\epsilon}(0)$ is obtained from (5) and (6).

Two important experiments connected with viscoelastic materials are the creep and relaxation tests. A creep test consists of applying a constant stress to an unstressed material and measuring the strain as a function of time. The function $\psi(t)$ which gives the strain response to a unit stress applied at t = 0 may be called the "creep function". For model A, (17) gives

$$\overline{\Psi}(s) = \left[\frac{1}{E_1}\left(s + \frac{E_1}{\eta_1}\right)\frac{1}{s} + \frac{1}{\eta_2}\,\frac{1}{\left(s + \frac{E_2}{\eta_2}\right)}\right]\frac{1}{s}, \tag{23}$$

where $\overline{\Psi}(s)$ is the Laplace transform of $\psi(t)$. Upon inversion,

$$\psi(t) = \frac{1}{E_1} + \frac{1}{\eta_1}t + \frac{1}{E_2}\left(1 - e^{-\frac{E_2}{\eta_2}t}\right). \tag{24}$$

A relaxation test consists of applying a constant strain to an unstrained model, and measuring the stress as a function of time. The stress response to a unit strain may be called the "relaxation function", $\phi(t)$. This function is obtained from (19) by first taking the Laplace transform of (19), and then inverting.

$$\overline{\phi}(s) = E_1'\,\frac{1}{\left(s + \frac{E_1'}{\eta_1'}\right)} + E_2'\,\frac{1}{\left(s + \frac{E_2'}{\eta_2'}\right)}; \tag{25}$$

$$\phi(t) = E_1' e^{-\frac{E_1'}{\eta_1'}t} + E_2' e^{-\frac{E_2'}{\eta_2'}t}. \tag{26}$$

The two expressions, $\phi(t)$ and $\psi(t)$, are not independent, but are related through their Laplace transforms. From (21)

(27) $$\overline{\Psi}(s) = \frac{s^2 + As + B}{Cs^2 + Gs}\,\frac{1}{s},$$

and

(28) $$\overline{\phi}(s) = \frac{Cs^2 + Gs}{s^2 + As + B}\,\frac{1}{s};$$

therefore,

$$\overline{\Psi}(s)\overline{\phi}(s) = \frac{1}{s^2},$$

and by the convolution associated with the Laplace transform,

$$\int_0^t \psi(t - y)\phi(y)\,dy = t.$$

If one notes that

(29) $$s\overline{\epsilon}(s) = L\left[\dot{\epsilon}(t)\right] + \epsilon(0),$$

and

(30) $$s\overline{\tau}(s) = L\left[\dot{\tau}(t)\right] + \tau(0),$$

then by means of the convolution integral, an integral form of the stress-strain relationship can be obtained:

(31) $$\epsilon(t) = \int_0^t \psi(t - y)\frac{d\,\tau(y)}{dy}\,dy + \psi(t)\,\tau(0),$$

and

(32) $$\tau(t) = \int_0^t \phi(t - y)\frac{d\,\epsilon(y)}{dy}\,dy + \phi(t)\,\epsilon(0).$$

If a better description of the creep and relaxation mechanisms is desired, additional Voigt elements can be added in series to model A, and additional Maxwell elements can be added in parallel with model B. The model composed of (n - 1) Voigt elements and 1 Maxwell element in series is called a generalized Voigt model; its creep response is given by

$$\psi(t) = \frac{1}{E_1} + \frac{1}{\eta_1}t + \sum_{i=2}^{n} \frac{1}{E_i}(1 - e^{-\frac{E_i}{\eta_i}t}). \tag{33}$$

The model composed of n parallel Maxwell elements is called a generalized Maxwell model; its relaxation response is

$$\phi(t) = \sum_{i=1}^{n} E_i' e^{-\frac{E_i}{\eta_i}t}. \tag{34}$$

Since the development of the differential expressions governing the generalized models parallels that of the four-parameter models, it is easy to see that (33) and (34) may also be used in (31) and (32).

Alfrey [1] states that the generalized Voigt model has an equivalent generalized Maxwell model. This fact is readily apparent if operator notation is used in writing the corresponding differential relationships. Suppose a generalized Voigt model is composed of 1 Maxwell element and n - 1 Voigt elements. The Maxwell element satisfies

$$(D + \frac{E_1}{\eta_1})\tau = E_1 D \epsilon_1,$$

and the i th Voigt element

$$\frac{1}{\eta_i}\tau = (D + \frac{E_i}{\eta_i})\,\epsilon_i .$$

The combined differential expression is

$$\left\{\prod_{i=1}^{n}(D + \frac{E_i}{\eta_i}) + \sum_{j=2}^{n}\left[\prod_{i=2}^{n}(D + \frac{E_i}{\eta_i})D\frac{E_1}{\eta_j}\right]\right\}\tau$$

(35)

$$= \prod_{i=2}^{n} E_1(D + \frac{E_i}{\eta_i})D\epsilon\ , \quad i \neq j.$$

On the other hand, a generalized Maxwell model composed of n parallel Maxwell elements, the i th element of which satisfies

$$(D + \frac{E_i'}{\eta_i'})\,\tau_i = E_i' D\epsilon ,$$

is governed by the differential equation

$$(36) \qquad \prod_{i=1}^{n}(D + \frac{E_i'}{\eta_i'})\tau = \sum_{j=1}^{n}\left[\prod_{i=1}^{n}(D + \frac{E_i'}{\eta_i'})E_j'\right]\epsilon \ .$$

Equations (35) and (36) are of the same order in the derivatives of τ and ϵ, and differ only in their corresponding coefficients. Hence, one set of constants, E_i', η_i', can be expressed in terms of the other, E_i, η_i, and vice-versa.

Now the generalized Voigt model has a discrete set of retardation times ξ_i, where $\xi_i = \frac{\eta_i}{E_i}$, and the Maxwell model has a discrete set of relaxation times $\xi_i' = \frac{\eta_i'}{E_i'}$. A further generalization of these models can be obtained by increasing the number of elements in such a way that the E_i becomes a continuous function of ξ_i, that is, $E = \frac{\eta(\xi)}{\xi}$ such that

$$(37) \qquad \psi(t) = \int_0^\infty \frac{1}{E(\xi)}(1 - e^{-\frac{t}{\xi}})d\xi + \frac{1}{E_1} + \frac{1}{\eta_1}t$$

exists; and E_i' becomes a continuous function of ξ_i', $E' = \frac{\eta'(\xi')}{\xi'}$, such that

$$(38) \qquad \phi(t) = \int_0^\infty E'(\xi')e^{-\frac{t}{\xi}}d\xi$$

exists. Additional information on continuous spectra can be found in Alfrey [1].

Besides giving an excellent intuitive notion of the mechanical behavior of a viscoelastic material, the four-parameter model also furnishes a means for approximating the behavior of more complicated models. For example, consider a generalized Voigt model which has a discrete set of retardation times. The retardation times can be divided into three categories according to the type of response most nearly typical of each category (elastic, delayed elastic, viscous flow). As a guide in making this separation one could use the duration of the experiment relative to the retardation times. The generalized Voigt model can now be replaced by a four-parameter model using one part of the model for each of the three divisions of the generalized model.

As an example, consider a generalized Voigt model which has the retardation times

$$\xi_2 = 4, \quad \xi_3 = 40, \quad \xi_4 = 100, \quad \xi_5 = 1000,$$

and elastic constants $E_i = 1$, $i = 1, 2, \cdots, 5$. In addition, suppose that the period of investigation is to be in the interval

$$t_1 \leq t \leq t_2, \; t_1 = 20, \; t_2 = 40.$$

The associated creep response is

$$\psi(t) = \frac{1}{E_1} + \frac{1}{\eta_1}t + \sum_{i=2}^{5} \frac{1}{E_i}(1 - e^{-\frac{t}{\xi_i}}),$$

but one would like to replace $\psi(t)$ by $\bar{\psi}(t)$, where

$$\bar{\psi}(t) = \frac{1}{E_a} + \frac{1}{\eta_a} + \frac{1}{E_b}(1 - e^{-\frac{t}{\xi_b}}).$$

The following criteria will be used to effect this simplification:

1) $\frac{1}{E_a}$ will include $\frac{1}{E_1}$ and all terms satisfying

$$e^{-\frac{t_3}{\xi_i}} < e^{-3} < 0.05, \; t_3 = \frac{1}{2}(t_1 + t_2);$$

the corresponding set of retardation times will be indexed by $i = 1, 2, 3, \cdots, j-1$.

2) $\frac{1}{\eta_a}$ will include $\frac{1}{\eta_1}$ and all other terms which satisfy the condition

$$1 - e^{-\frac{t}{\xi_i}} = \frac{1}{\xi_i}t + R_n,$$

where $R_n < \frac{1}{2\xi_i^2}t_2^2 < e^{-3}$ with the accompanying set of retardation times indexed by $i = m, m+1, \cdots, n$.

3) the remaining terms will be combined into a single Voigt element where

$$\frac{1}{E_b} = \sum_{i=j}^{m-1} \frac{1}{E_i},$$

and ξ_b satisfies the equation

$$\sum_{i=j}^{m-1} \frac{1}{E_i}\left(e^{-\frac{t_1}{\xi_i}} + e^{-\frac{t_2}{\xi_i}}\right) = \frac{1}{E_b}\left(e^{-\frac{t_1}{\xi_b}} + e^{-\frac{t_2}{\xi_b}}\right).$$

This definition of ξ_b forces the difference between

$$\sum_{i=j}^{m-1} \frac{1}{E_i}\left(1 - e^{-\frac{t}{\xi_i}}\right)$$

$$\frac{1}{E_b}\left(1 - e^{-\frac{t}{\xi_b}}\right)$$

to be the same at t_1 and t_2.

For $\xi_2 = 4$, $t_3 = 30$; hence, $e^{-\frac{30}{4}} \sim .001 < e^{-3}$. Therefore, $\frac{1}{E_2} \sim \frac{1}{E_2}\left(1 - e^{-\frac{t}{\xi_2}}\right)$, and $\frac{1}{E_a} = \frac{1}{E_1} + \frac{1}{E_2}$.

For $\xi_5 = 1000$, $1 - e^{-\frac{t_2}{1000}} = 0.001t_2 + R_n$, where $R_n = 8 \times 10^{-4} < 10^{-3} < e^{-3}$. Hence, $\frac{1}{\eta_a} = \frac{1}{\eta_1} + \frac{1}{\eta_5}$.

Finally ξ_3 and ξ_4 can be combined in ξ_b with the result that

$$\frac{1}{E_b} = \frac{1}{E_3} + \frac{1}{E_4}, \quad \xi_b \sim 60, \quad E_b = \frac{1}{2}.$$

Consequently

$$\Psi(t) = 2 + \left(\frac{1}{\eta_1} + \frac{1}{\eta_5}\right)t + 2\left(1 - e^{-\frac{t}{60}}\right).$$

Another approach to setting up the equations of the phenomenological approach is based on the Boltzmann Superposition Principle [9] , an excellent account of which can be found in Leaderman [10] . Disregarding the effect of continuous flow the superposition principle embodies two assumptions:

1) the deformation is composed of two separate mechanisms: an instantaneous response, and a delayed response which depends upon the previous loading history;

2) the total deformation due to a complex loading history is the simple summation of the deformations due to the separate loading effects.

For a one-dimensional model which also exhibits flow, this principle leads to the relationship

$$(39) \quad \tau(t) = E\epsilon(t) + \int_{-\infty}^{t} \left[\eta\delta(t-y) + \Phi(t-y)\right] \frac{d\,\epsilon(y)}{dy}dy.$$

The details of this development can be found in Gross [3] , Leaderman [10] , and Volterra [11] . Volterra [12] has shown that if linear superposition is true for relaxation it is also true for creep; hence

$$(40) \qquad \epsilon(t) = \frac{1}{E}\mathcal{T}(t) + \int_{-\infty}^{t}\left[\frac{1}{\eta}(t-y) + \Psi(t-y)\right]\frac{d\mathcal{T}(y)}{dy}dy.$$

The above integrals $\left(\int_{-\infty}^{t}[\quad]dy\right)$ are Stieltjes integrals; thus they need not be zero even though the prescribed deformation or stress $\epsilon(t)$ or $\mathcal{T}(t)$ may have a constant value except for a jump at the instant of application. The lower limit of integration is $(-\infty)$ to indicate that the integration must begin when the body is first disturbed. If for $t < 0$ the body remains unperturbed, then the lower limit may be replaced by zero.

Furthermore, $\Psi(t - y)$ and $\Phi(t - y)$ are defined as follows:

$\Psi(t - y)$ is a continuous non-decreasing function such that $\Psi(t - y) = 0$ when $t \leq y$ and $\lim_{t\to\infty} \Psi(t - y)$ exists;

$\Phi(t - y)$ is a continuous non-increasing function such that $\Phi(t - y) = 0$ when $t < y$, $\Phi(0)$ exists, and $\lim_{t\to\infty}\Phi(t - y) = 0$.

It should be noted that (31), (32) and (39), (40) are equivalent though their derivations are somewhat different. This correspondence can be effected by defining Ψ and Φ as follows:

$$\begin{aligned} &\Psi(t - y) = \psi(t - y) - \frac{1}{\eta}(t - y) - \frac{1}{E}, \\ (41) \qquad &\Phi(t - y) = \phi(t - y), \quad \eta = 0, \; E = 0. \end{aligned}$$

When ψ and Ψ are represented by mechanical models, ψ has an additional Maxwell element in series with the model depicting $\bar{\Psi}$. Though both ϕ and Φ are represented by the same model, the model which depicts the behavior of (39) has a Voigt element in parallel with the model describing (32).

Equations (41) can now be utilized to relate (31), (32) with (39), (40). The equivalence of (31) and (40) will now be demonstrated. Since $\mathcal{T}(t)$ is continuous except for a possible finite jump at t = 0 (the deformation is applied at t = 0 in the first pair and at t = 0^+ in the second), it can be reduced to the form

$$\epsilon(t) = \frac{1}{E}\mathcal{T}(t) + \left[\psi(0) - \frac{1}{E}\right]\mathcal{T}(0) + \int_{0^+}^{t}\left[\psi(t - y) - \frac{1}{E}\right]\frac{d\,\mathcal{T}(y)}{dy}dy,$$

where the integral $\int_{0^+}^{t}(\ )dy$ is a Riemann integral. Therefore,

$$\epsilon(t) = \frac{1}{E}\mathcal{T}(t) + \psi(0)\,\mathcal{T}(0) - \frac{1}{E}\mathcal{T}(0) - \frac{1}{E}\left[\mathcal{T}(t) - \mathcal{T}(0)\right] + \int_{0^+}^{t}\psi(t - y)\frac{d\,\mathcal{T}(y)}{dy}dy$$

which is identical with (31). The correspondence between (32) and (40) can be shown in a similar manner.

It has already been noted that a Voigt material does not respond instantaneously to a stress stimulus, but shows a delayed strain effect which continues after the stress is removed. The term "memory effect" refers to the apparent ability of a material to remember its past stress history, while the "recollection effect" refers to the apparent ability of a material to recall its past strain history.

A striking example of the memory effect can be obtained by observing the strain response of a double Voigt model for which $\xi_1 < \xi_2$, where $\xi_1 = \frac{\eta_1}{E_1}$, $\xi_2 = \frac{\eta_2}{E_2}$, to the stress history

$$\tau = \begin{cases} \tau_0, & 0 \leq t < a, \\ -\tau_0, & a \leq t < a+b, \\ 0, & t \geq a+b, \end{cases}$$

where $a > 0$, $b > 0$, τ_0 = constant. The strain response is

$$\epsilon(t) = \tau_0 \left[\frac{1}{E_1}(1 - e^{-\frac{t}{\xi_1}}) + \frac{1}{E_2}(1 - e^{-\frac{t}{\xi_2}}) \right], \quad 0 \leq t < a;$$

$$(42) \quad \epsilon(t) = \tau_0 \left[-\frac{1}{E_1} - \frac{1}{E_2} + \frac{1}{E_1}(2e^{\frac{t}{\xi_1}} - 1)e^{-\frac{t}{\xi_1}} + \frac{1}{E_2}(2e^{\frac{t}{\xi_2}} - 1)e^{-\frac{t}{\xi_2}} \right], \quad a \leq t < a+b;$$

$$\epsilon(t) = \tau_0 \left[\frac{1}{E_1}(2e^{\frac{a}{\xi_1}} - e^{\frac{a+b}{\xi_1}} - 1)e^{-\frac{t}{\xi_1}} + \frac{1}{E_2}(2e^{\frac{a}{\xi_2}} - e^{\frac{a+b}{\xi_2}} - 1)e^{-\frac{t}{\xi_2}} \right], \quad t \geq a+b.$$

Let $\zeta_1 = \frac{1}{\xi_1}$, $\zeta_2 = \frac{1}{\xi_2}$, $\zeta_1 > \zeta_2$,

$$A = 2e^{\zeta_1 a} - e^{\zeta_1 (a+b)} - 1,$$

$$B = 2e^{\zeta_2 a} - e^{\zeta_2 (a+b)} - 1;$$

then, the last expression in (42) has the form

$$\epsilon(t) = \tau_0\left(\frac{A}{E_1}e^{-\zeta_1 t} + \frac{B}{E_2}e^{-\zeta_2 t}\right), \quad t \geq a + b,$$

where $\zeta_1 > 0$, $\zeta_2 > 0$.

The dependence of the strain on the previous stress history is quite perceptible in the time interval $t \geq a + b$. In fact, the strain response is determined by a and b since ζ_1 and ζ_2 are already prescribed, and A, B are defined in terms of a, b. When $A < 0$ and $B > 0$, there exists an $a > 0$ and an interval $b_1 < b_3 < b < b_2$ for which the strain changes from negative to positive values before decaying to zero; but if $A < 0$ and $B < 0$, the strain always returns to zero through negative values.

Clearly $A < 0$, $B < 0$ requires $\epsilon(t) < 0$ for all $t \geq a + b$; in addition, $\lim_{t \to \infty} \epsilon(t) = 0$. The truth of the first assertion, however, is not quite so obvious. Let b_1 be the zero of $A = 0$, and b_2 the zero of $B = 0$, then

$$(43) \qquad \begin{aligned} A &= 2e^{\zeta_1 a} - e^{\zeta_1 (a+b)} - 1 < 0, \ b \geq b_1, \\ B &= 2e^{\zeta_2 a} - e^{\zeta_2 (a+b)} - 1 > 0, \ 0 < b < b_2. \end{aligned}$$

The existence and uniqueness of b_1 and b_2 is assured since both A and B are continuous monotone decreasing functions of b having positive least upper bounds and no lower bounds.

By the definitions of b_1 and b_2

$$e^{\zeta_2 b_2} = 2 - e^{-\zeta_2 a},$$

and

$$e^{\zeta_1 b_1} = 2 - e^{-\zeta_1 a},$$

Hence,

$$\lim_{a\to\infty} \zeta_2 b_2 = \lim_{a\to\infty} \zeta_1 b_2 = \ln 2,$$

and

$$\lim_{a\to\infty} b_2 = \frac{\ln 2}{\zeta_2}, \lim_{a\to\infty} b_1 = \frac{\ln 2}{\zeta_1};$$

thus

$$\lim_{a\to\infty} (b_2 - b_1) = (\xi_2 - \xi_1)\ln 2 > 0.$$

Therefore, there exists an $a_1 > 0$ such that for every $a > a_1$, $b_2 - b_1 > 0$; that is, there exists an interval $b_1 < b < b_2$ for which (43) holds.

In passing, it should be mentioned that $b_2 < a$, for if this condition were not satisfied, b_2 could not satisfy its defining equation.

Since A and B are continuous monotone decreasing functions of b, there exists an interval $b_1 < b_3 < b < b_2$, for which $A < 0$, $B > 0$ and $\epsilon(a + b) < 0$. With b restricted to this interval it is always possible to find a $t > 0$ such that

$\epsilon(t) > 0$. Let $\zeta_1 = \zeta_2 + d$; then

$$\epsilon(t) = \mathcal{T}_0 \frac{B}{E_2}\left(\frac{A}{B}\frac{E_2}{E_1} + e^{dt}\right)e^{-\zeta_1 t}, \quad t \geq a + b;$$

but $e^{dt} > \frac{|A|}{B}\frac{E_2}{E_1}$ whenever $t > \frac{1}{d}\log\frac{|A|}{B}\frac{E_2}{E_1}$. Therefore, $\epsilon(t) > 0$ whenever $t > \frac{1}{d}\log\frac{|A|}{B}\frac{E_2}{E_1}$. The conditions $A < 0$, $B > 0$ are not sufficient since $\epsilon(a + b) > 0$ when $b_1 < b < b_3$. The behavior described above is depicted in Figure 2.

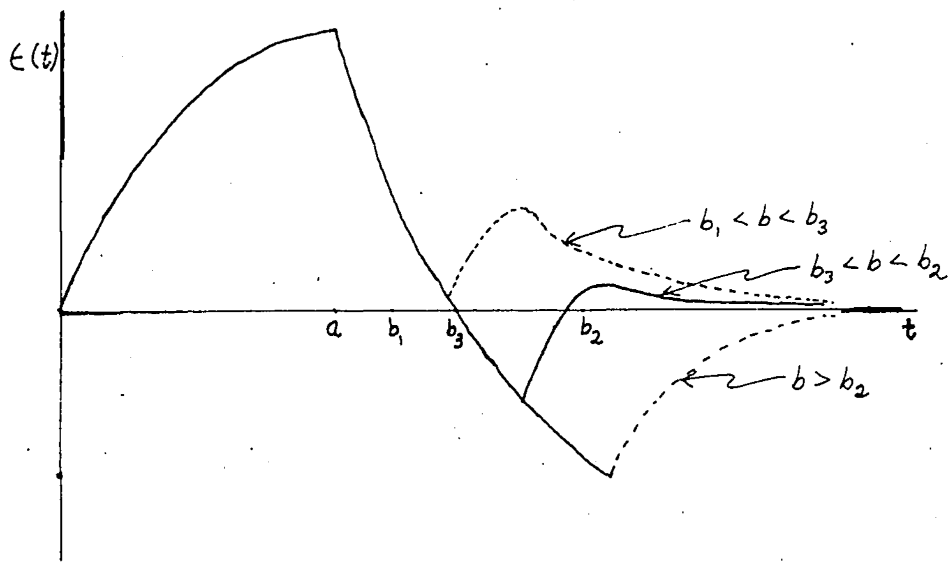


Figure 2. Memory effect for a double Voigt model

So far only one-dimensional behavior of viscoelastic materials has been discussed. For linear, homogeneous, isotropic, viscoelastic media the simplest possible representation of their phenomenological behavior in three dimensions must either have two separate relations connecting the stresses and strains, or three distinct operators in the stress-strain relations. Using the information obtained for one-dimensional viscoelasticity, two separate generalized models composed of springs and dashpots can be constructed; one for response to shear, the other for response to normal compression. This operator classification is quite convenient because the operators can be measured experimentally. Let[1]

$$e_{ij} = \epsilon_{ij} - \frac{1}{3}\epsilon_{kk}\delta_{ij},$$

and

$$S_{ij} = \tau_{ij} - \frac{1}{3}\tau_{kk}\delta_{ij}.$$

In terms of S_{ij} and e_{ij} the time dependent stress-strain law can be written as

$$\begin{aligned} PS_{ij} &= Qe_{ij}, \\ P'\tau_{ii} &= Q'\tau_{ii}, \end{aligned} \tag{44}$$

where

$$P = \sum_{i=0}^{n} p_i \frac{\partial^i}{\partial t^i}, \qquad Q = \sum_{i=0}^{m} q_i \frac{\partial^i}{\partial t^i},$$

$$P' = \sum_{i=0}^{n'} p'_i \frac{\partial^i}{\partial t^i}, \qquad Q' = \sum_{i=0}^{m'} q'_i \frac{\partial^i}{\partial t^i},$$

---

[1]From this point on the summation convention is used. Repeated Latin subscripts are summed from one to three. Repeated Greek subscripts are summed from one to two. If any subscript is not repeated, it represents all terms in its range, i.e., Latin indices from one to three, Greek indices from one to two.

and p_i, q_i, p_i', q_i' are all constants. Moreover, it should be noted that the operators P, P', Q, and Q' are linear operators, and all products of operators are commutative. That the two relations (44) so defined are independent is easily verified. Since

$$P(\mathcal{T}_{ij} - \tfrac{1}{3}\mathcal{T}_{kk}\delta_{ij}) = Q(\epsilon_{ij} - \tfrac{1}{3}\epsilon_{kk}\delta_{ij}),$$

by contracting the tensors

$$P(\mathcal{T}_{ii} - \mathcal{T}_{kk}) = Q(\epsilon_{ii} - \epsilon_{kk}),$$

and thus the second of (44) can not in general be obtained from the first.

Often a viscoelastic problem can be simplified considerably since it has been shown experimentally that many viscoelastic materials are elastic under normal stress ($P' = 1$, $Q' = H$, H a constant).

Another form of the stress-strain law can be obtained from (44) by forming the combination

$$\begin{aligned} P'P(S_{ij} + \tfrac{1}{3}\mathcal{T}_{kk}\delta_{ij}) &= P'Q(e_{ij} + \tfrac{1}{3}\epsilon_{kk}\delta_{ij}) \\ &\quad + \tfrac{1}{3}(PQ' - P'Q)\epsilon_{kk}\delta_{ij}. \end{aligned} \tag{45}$$

By virtue of the definitions of S_{ij} and e_{ij} this implies

$$P'P\,\mathcal{T}_{ij} = P'Q\,\epsilon_{ij} + \tfrac{1}{3}(PQ' - P'Q)\,\epsilon_{kk}\delta_{ij}. \tag{46}$$

In turn, for linear, homogeneous, isotropic media, (46) can be related to the ordinary elastic stress-strain law

(47) $$\mathcal{T}_{ij} = 2\mu\,\epsilon_{ij} + \lambda\,\epsilon_{kk}\delta_{ij}$$

by symbolically setting

$$2\mu = \frac{Q}{P} \text{ and } \lambda = \frac{1}{3}\left(\frac{Q'}{P'} - \frac{Q}{P}\right).$$

By means of this correspondence solutions for many viscoelastic problems can be obtained directly from the existing elastic solutions. The quotients $\frac{Q}{P}$, $\frac{Q'}{P'}$ do not represent a new operator, rather they act as an intermediate step in relating to a corresponding elastic problem.

Applying the Laplace transform to (44)

(48) $$\overline{P}\overline{S}_{ij} = \overline{Q}\overline{e}_{ij}, \qquad \overline{P'}\,\overline{\mathcal{T}}_{ii} = \overline{Q'}\,\overline{\epsilon}_{ii},$$

where

$$\overline{P} = \sum_{i=0}^{n} p_i s^i, \quad \overline{Q} = \sum_{i=0}^{m} q_i s^i,$$

$$\overline{P'} = \sum_{i=0}^{n'} p_i' s^i, \quad \overline{Q'} = \sum_{i=0}^{m'} q_i' s^i,$$

$$\overline{S_{ij}} = LS_{ij}, \quad \overline{e_{ij}} = Le_{ij}, \text{ etc.}$$

No initial conditions occur in the transform because the operator pairs P, Q; P', Q' each correspond to some generalized model, and the prescribed initial elastic terms cancel in the transformed problem.

Combining the expressions in (48),

(49) $$\overline{P}\overline{P'}\,\overline{\mathcal{T}}_{ij} = \overline{P'}\,\overline{Q}\,\overline{\epsilon}_{ij} + \frac{1}{3}(\overline{P}\overline{Q'} - \overline{P'}\overline{Q})\,\overline{\epsilon}_{kk}\delta_{ij}.$$

Dividing by $\overline{P}\overline{P'}$,

$$(50)\qquad \overline{\tau}_{ij} = \frac{\overline{Q}}{\overline{P}}\overline{\epsilon}_{ij} + \frac{1}{3}\left(\frac{\overline{Q'}}{\overline{P'}} - \frac{\overline{Q}}{\overline{P}}\right)\overline{\epsilon}_{kk}\delta_{ij},$$

where now one can make the correspondence

$$2\overline{\mu} = \frac{\overline{Q}}{\overline{P}},\quad \overline{\lambda} = \frac{1}{3}\left(\frac{\overline{Q'}}{\overline{P'}} - \frac{\overline{Q}}{\overline{P}}\right).$$

Thus, $\overline{\mu}$ and $\overline{\lambda}$ play the same role in the transformed viscoelastic problems as μ and λ in the elastic problem.

Since the transform of (46) must be identical with (49), the initial elastic terms must cancel in the transform of (46).

By a similar development the strains can be found in terms of the stresses. Omitting the details, the transformed relation reduces to

$$(51)\qquad \overline{\epsilon}_{ij} = \frac{\overline{P}}{\overline{Q}}\overline{\tau}_{ij} + \frac{1}{3}\left(\frac{\overline{P'}}{\overline{Q'}} - \frac{\overline{P}}{\overline{Q}}\right)\overline{\tau}_{kk}\delta_{ij}.$$

The following equations can then be obtained from (50) and (51) by convolution:

$$(52)\qquad \tau_{ij} = \phi_1(t)\,\epsilon_{ij}(0) + \phi_2(t)\,\epsilon_{kk}(0)\delta_{ij} + \int_0^t\left[\phi_1(t-y)\frac{\partial\epsilon_{ij}(y)}{\partial y} + \phi_2(t-y)\frac{\partial\epsilon_{kk}(y)}{\partial y}\delta_{ij}\right]dy,$$

where

$$\phi_1(t) = L^{-1}\left(\frac{\overline{Q}}{\overline{P}}\frac{1}{s}\right),\quad \phi_2(t) = L^{-1}\left[\frac{1}{3}\left(\frac{\overline{Q'}}{\overline{P'}} - \frac{\overline{Q}}{\overline{P}}\right)\right];$$

and

$$\epsilon_{ij} = \psi_1(t)\, \mathcal{T}_{ij} + \psi_2(t)\, \mathcal{T}_{kk}(0)\delta_{ij} + \int_0^t \left[\psi_1(t-y)\frac{\partial \mathcal{T}_{ij}(y)}{\partial y} + \psi_2(t-y)\frac{\partial \mathcal{T}_{kk}(y)}{\partial y}\delta_{ij} \right] dy, \tag{53}$$

where

$$\psi_1(t) = L^{-1}\left[\frac{\overline{P}}{\overline{Q}}\frac{1}{s} \right], \quad \psi_2(t) = L^{-1}\left[\frac{1}{3s}\left(\frac{\overline{P'}}{\overline{Q'}} - \frac{\overline{P}}{\overline{Q}}\right) \right].$$

The functions $\psi_1(t)$ and $\phi_1(t)$ are analogous to the memory and recollection functions defined in the one-dimensional theory. $\phi_1(t)$ represents a response to a unit shear strain; $\psi_1(t)$ the response to a unit shear stress.

Just as in the one dimensional theory, a set of functions Φ_1, Φ_2, Ψ_1, Ψ_2 can be defined which are related to ϕ_1, ϕ_2, ψ_1, ψ_2 in the same way as Φ, Ψ is related to ϕ, ψ. Using this set of functions together with the Stieltjes integral, (52), (53) can be replaced by (54), (55). The latter pair of integrals is related to the former as (39), (40) is related to (32), (31).

$$\mathcal{T}_{ij} = 2\mu\, \epsilon_{ij} + \lambda\, \epsilon_{kk}\delta_{ij} + \int_{-\infty}^t \left[2\eta \frac{\partial \epsilon_{ij}(y)}{\partial y} + \lambda_1 \frac{\partial \epsilon_{kk}(y)}{\partial y}\delta_{ij} \right] \delta(t-y)\,dy + \int_{-\infty}^t \left[\Phi_1(t-y)\frac{\partial \epsilon_{ij}(y)}{\partial y} + \Phi_2(t-y)\frac{\partial \epsilon_{kk}(y)}{\partial y}\delta_{ij} \right] dy; \tag{54}$$

and

$$\epsilon_{ij} = \frac{1}{2\mu}\mathcal{T}_{ij} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}\mathcal{T}_{kk}\delta_{ij}$$

$$+ \int_{-\infty}^{t}\left[\frac{1}{2\eta}\frac{\partial \mathcal{T}_{ij}(y)}{\partial y} - \frac{\lambda_1}{2\eta(3\lambda_1+2\eta)}\frac{\partial \mathcal{T}_{kk}(y)}{\partial y}\delta_{ij}\right](t-y)\,dy \tag{55}$$

$$+ \int_{-\infty}^{t}\left[\Psi_1(t-y)\frac{\partial \mathcal{T}_{ij}(y)}{\partial y} + \Psi_2(t-y)\frac{\partial \mathcal{T}_{kk}(y)}{\partial y}\delta_{ij}\right]dy.$$

An integral representation given by Volterra [11] for linear homogeneous anisotropic viscoelastic material is

$$\mathcal{T}_{ij} = c_{ijk\ell}\,\epsilon_{k\ell} + \int_{-\infty}^{t}\left[d_{ijk\ell}\delta(t-y) + \Phi_{ijk\ell}(t-y)\right]\frac{\partial \epsilon_{k\ell}(y)}{\partial y}dy, \tag{56}$$

and

$$\epsilon_{ij} = C_{ijk\ell}\,\epsilon_{k\ell} + \int_{-\infty}^{t}\left[D_{ijk\ell}(t-y) + \Psi_{ijk\ell}(t-y)\right]\frac{\partial \mathcal{T}_{k\ell}(y)}{\partial y}dy. \tag{57}$$

As in the one-dimensional case ϕ_1, ϕ_2, ψ_1, ψ_2 can be extended to include continuous spectra, thereby obtaining a more general description of the phenomenon.

Another stress-strain relation is sometimes used, namely, the dynamic modulus representation, but it will not be discussed in this paper.

FORMULATION OF THE VISCOELASTIC PROBLEM

Let every point of a continuous three-dimensional body, at rest for $t < 0$, be referred to a rectangular Cartesian coordinate system A_i. The coordinates of a point P_0 in the unstrained state are a_i. Let P_0' be an adjacent point with coordinates $a_i + da_i$; then the element of arc length connecting P_0 and P_0' is

(58) $$dS_0^2 = da_i da_i .$$

Suppose that at time $t > 0$ the body is referred to a new rectangular Cartesian coordinate system X_i parallel to the A_i system. Let the coordinates, in the strained state, of the point P, previously at P_0 in the unstrained state, be x_i. Let P' be a neighboring point in the strained state having coordinates $x_i + dx_i$, then the element of arc length connecting P and P' is given by

(59) $$dS^2 = dx_i dx_i .$$

For physical reasons the new coordinates x_i of the point P must be continuous functions of the old coordinates a_i of the point P_0, and the time t; that is,

$$x_i = f_i(a_1, a_2, a_3, t).$$

The coordinates of the point P can be written in terms of the initial coordinates and time in the following manner

(60) $$x_i = a_i + u_i(a_1, a_2, a_3, t),$$

where the u_i are sufficiently small, and where $u_i(a_1, a_2, a_3, t)$ are the components of the displacement vector of the point P relative to P_0.

If P', having coordinates $x_i + dx_i$, is the new position of P_0', then the relation between dx_i and da_i is

(61)
$$dx_i = da_i + u_{i,j}da_j,$$

where
$$,j \equiv \frac{\partial}{\partial a_j}.$$

Thus,

(62)
$$dx_s dx_s - da_s da_s = 2\gamma^*_{ij} da_i da_j,$$

where

(63)
$$\gamma^*_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) + \frac{1}{2}u_{s,i}u_{s,j}$$

are the components of the strain tensor.

Now the components of an infinitesmal rotation tensor are given by

$$w_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}).$$

Hence, if the strain γ^*_{ij} is to be a pure strain without rotation

$$u_{i,j} = u_{j,i}.$$

Let γ_{ij} be the components of the pure strain tensor; then

(64)
$$\gamma_{ij}(a_1, a_2, a_3, t) = \epsilon_{ij}(a_1, a_2, a_3, t) + \alpha_{ij}(a_1, a_2, a_3, t),$$

where

(65)
$$\epsilon_{ij}(a_1, a_2, a_3, t) = \frac{1}{2}\left[u_{i,j}(a_1, a_2, a_3, t) + \right.$$

$$u_{j,i}(a_1,\ a_2,\ a_3,\ t)\Big],$$

$$\alpha_{ij}(a_1,\ a_2,\ a_3,\ t) = \tfrac{1}{2}u_{i,s}(a_1,\ a_2,\ a_3,\ t)u_{j,s}(a_1,a_2,a_3,t).$$

Hereafter the argument will be dropped from the notation, but the dependence on the original coordinates and time will be presumed unless otherwise stated.

Rivlin [13] has shown that a general relationship between the stress referred to the coordinate system X_i and the strain referred to the coordinate system A_i can be obtained through a potential function W. When the medium is isotropic, W depends on three invariants of the strain I_1, I_2, I_3, and scalar functions of the original coordinates a_i. If the scalar functions are constants, then the medium is called homogeneous. Of course, I_1, I_2, and I_3 are functions of both the original and final coordinates, hence they can be considered alternatively as functions of the final coordinates and time.

Suppose a homogeneous isotropic material, initially referred to a rectangular Cartesian coordinate system A_i fixed in the body is deformed, and is now referred to a rectangular Cartesian coordinate system X_i which has the position previously occupied by the system A_i (A_i may no longer be a rectangular Cartesian coordinate system). The stress-strain law is then

$$(66)\quad \sigma_{ij} = \frac{2}{\beta}\left[g_{ij}\frac{\partial W}{\partial I_1} - G_{ij}\frac{\partial W}{\partial I_2} + \left(I_3\frac{\partial W}{\partial I_3} + I_2\frac{\partial W}{\partial I_2}\right)\delta_{ij}\right],$$

where

$$(67)\qquad \begin{aligned} g_{ij} &= (\delta_{is} + u_{i,s})(\delta_{js} + u_{j,s}), \\ G_{ij} &= \text{cofactor of } g_{ij} \text{ in } \det g_{ij}, \\ I_1 &= g_{ss},\ I_2 = G_{ss},\ I_3 = \det g_{ij} \\ \beta^2 &= I_3. \end{aligned}$$

A viscoelastic stress-strain relation can be obtained from the elastic case by replacing the elastic constants by differential operators $\frac{P}{Q}$ as defined in (44), and thereby forming equations similar to (46).

Relative to the coordinate system X_i the equilibrium equations are

$$(68)\qquad \sigma_{ji/j}(x_1, x_2, x_3, t) + \rho F_i(x_1, x_2, x_3, t) = \rho\frac{\partial^2 u_i}{\partial t^2}(x_1, x_2, x_3, t),$$

where

$$/j = \frac{\partial}{\partial x_j} = \frac{1}{\beta}\,\frac{\partial\beta}{\partial(u_{j,s})}\,\frac{\partial}{\partial a_s},$$

ρ_0 = density of the undeformed body and

$$\rho = \frac{\rho_0}{\beta}.$$

Since the strain is defined with respect to the undeformed system A_i, it is necessary that the equilibrium equations also be written in this system. These then take the form

$$(69)\qquad \frac{\partial\beta}{\partial u_{j,s}}\,\sigma_{ji,s} + \rho_0 F_i = \rho_0\frac{\partial^2 u_i}{\partial t^2}.$$

The compatibility equations for the strains referred to A_i are

$$\gamma_{i\ell,jk} + \gamma_{jk,i\ell} - \gamma_{ik,j\ell} - \gamma_{j\ell,ik} + 4(2\gamma_{mn} + \delta_{mn})(\gamma_{i\ell m}\gamma_{jkn} - \gamma_{ikm}\gamma_{j\ell n}) = 0 \tag{70}$$

where

$$\gamma_{i\ell m} = \gamma_{mi,\ell} + \gamma_{m\ell,i} - \gamma_{i\ell,m}.$$

To complete the statement of a rheological problem a set of boundary conditions must be prescribed. These conditions take the form of displacement boundary conditions or stress boundary conditions. If displacement boundary conditions are prescribed, they are given with respect to initial coordinates a_i and time t, hence require no alteration in the statement of a second order problem:

$$u_i = f_i(a_1,\ a_2,\ a_3,\ t). \tag{71}$$

On the other hand, stress boundary conditions are expressed in final coordinates, but with respect to the unstrained body. Consequently the stresses, surface normals, and final surface areas must be expressed in terms of the original coordinates a_i:

$$T_i = \frac{\partial\beta}{\partial(u_{k,s})}\sigma_{ik}n_s. \tag{72}$$

Equations (63), (66), (69), (70), (71), and (72) completely specify the b. v. problem, but if any problems are to be solved some assumptions will have to be made. To this end the following assumptions are proposed:

1) the undeformed state is also an unstressed state;

2) the medium is homogeneous, isotropic, and can be described by three viscoelastic operators;

3) third degree displacement gradients can be ignored relative to second degree displacement gradients, i.e.,

$$\left[\frac{\partial u_i}{\partial a_j}\right]^3 < < \left[\frac{\partial u_i}{\partial a_j}\right]^2 .$$

These conditions modify both the stress-strain relations and the equilibrium equations. According to elastic theory, if W contains only terms of second degree in the displacement gradients,

(73) $$W = a_1 J_2 + a_2 J_1,$$

where

$$J_1 = I_1 - 3,\ J_2 = I_2 - 2I_1 + 3,\ J_3 = I_3 - I_2 + I_1 - 1.$$

The stress-strain rule then takes the form

(74) $$\sigma_{ij} = 2\mu \epsilon_{ij} + \lambda \epsilon_{ss}\delta_{ij} + (2\mu\alpha_{ij} + \lambda\alpha_{ss}\delta_{ij})$$
$$+ \left[2(\lambda + \mu)\epsilon_{ij} - \lambda \epsilon_{ss}\delta_{ij}\right]\epsilon_{ss}$$
$$+ 4\mu(E_{ij} - E_{ss}\delta_{ij}),$$

where

$$\alpha_{ij} = \frac{1}{2} u_{i,s}u_{j,s},$$

$$E_{ij} = \text{Cof } \epsilon_{ij} \text{ in det } \epsilon_{ij},$$

$$a_1 = -\frac{\mu}{2}, \quad a_2 = \frac{\lambda + 2\mu}{8}.$$

Replacing μ and λ by viscoelastic operators as suggested in (47), the second order viscoelastic stress-strain law is

$$\begin{aligned} R\sigma_{ij} = {} & 2M\epsilon_{ij} + N\epsilon_{ss}\delta_{ij} \\ & + 2M\alpha_{ij} + N\alpha_{ss}\delta_{ij} \\ & + [2(M+N)\epsilon_{ij} - N\epsilon_{ss}\delta_{ij}]\epsilon_{ss} \\ & + 4M(E_{ij} - E_{ss}\delta_{ij}), \end{aligned} \tag{75}$$

where

$$R = PP', \quad M = \frac{1}{2}P'Q, \quad N = \frac{1}{3}(PQ' - QP').$$

The stress σ_{ij} can be decomposed into first and second order stresses σ'_{ij} and σ''_{ij} respectively:

$$R\sigma_{ij} = R\sigma'_{ij} + R\sigma''_{ij}, \tag{76}$$

where

$$R\sigma'_{ij} = 2M\epsilon_{ij} + N\epsilon_{ss}\delta_{ij},$$

and

$$\begin{aligned} R\sigma''_{ij} = {} & 2M\alpha_{ij} + N\alpha_{ss}\delta_{ij} \\ & + [2(M+N)\epsilon_{ij} - N\epsilon_{ss}\delta_{ij}]\epsilon_{ss} \\ & + 4M(E_{ij} - E_{ss}\delta_{ij}). \end{aligned}$$

The usefulness of this separation will become evident in the modified equilibrium equations.

A set of initial conditions must still be prescribed in order to complete the statement of the viscoelastic problem.

In the linear viscoelastic problem the initial elastic terms cancelled out in the transformed stress-strain law. This condition will now be extended to the second order stress-strain law by requiring the initial terms to cancel in the transformed problem, and to assume the value of the elastic solution wherever they need be specified.

Finally, then, the modified equilibrium equations are

$$\left[(1 + \epsilon_{ss})\delta_{jk} - u_{j,k}\right] \sigma'_{ki,j} + \sigma''_{ki,k} + \rho_0 F_i = \rho_0 \frac{\partial^2 u_i}{\partial t^2}, \tag{77}$$

and the stress boundary conditions are

$$T_i = \left[(1 + \epsilon_{ss})\delta_{jk} - u_{j,k}\right] \sigma'_{ki} n_j + \sigma''_{ki} n_k. \tag{78}$$

In summary, the modified second order viscoelastic problem can be formulated as follows:

Determine the stresses and displacements which satisfy (63), (70), (75), and (77) in the body; the boundary conditions (71) and (78) wherever they are prescribed; and a set of initial conditions which furnish terms that cancel in the Laplace transform of (75).

THE SOLUTION OF LINEAR VISCOELASTIC PROBLEMS

Suppose that the viscoelastic properties of a material can be expressed by (48), and that the boundary conditions are amenable to the Laplace transform; then the method of Lee [4] can be employed to solve this linear problem. In this method, the Laplace transform is applied to every expression involved in order to obtain an equivalent elastic problem. The modified solution is then inverted to obtain the viscoelastic solution.

In practice, the modified elastic solution contains the transformed boundary conditions together with certain combinations of viscoelastic operators. These transformed elastic operators can be represented symbolically by their corresponding elastic constants such as $\overline{\mu}$, $\overline{K}$, etc. If the boundary conditions are not time dependent, then their transforms are merely a constant times $\frac{1}{s}$. For most problems the inversion can be obtained through the convolution of the modified elastic operators and the boundary conditions.

Equations (48) stipulate that the material can be represented by two generalized models; one for response in shear, the other for response in dilatation. The choice of the Maxwell or Voigt representation of the same model depends, of course, on the particular problem. Thus if the displacements are prescribed, the generalized Maxwell model gives the better description; whereas, if the stresses are prescribed, the generalized Voigt model gives both the simpler mathematical description and the better intuitive notion of the behavior.

The transformed elastic operator associated with shear is

$$2\bar{\mu} = \frac{\bar{Q}}{\bar{P}}; \tag{79}$$

the transformed elastic operator associated with dilatation is

$$3\bar{K} = \frac{\overline{Q'}}{\overline{P'}}. \tag{80}$$

By taking a specific linear combination of (79) and (80) a third transformed elastic operator $\bar{\lambda}$ can be formed

$$\bar{\lambda} = \frac{1}{3}\left(\frac{\overline{Q'}}{\overline{P'}} - \frac{\bar{Q}}{\bar{P}}\right). \tag{81}$$

Since

$$\overline{S_{ij}} = 2\bar{\mu}\,\overline{\epsilon_{ij}}, \tag{82}$$

and

$$\overline{\tau_{ii}} = 3\bar{K}\,\overline{\epsilon_{ii}}; \tag{83}$$

the quantities

$$L^{-1}(2\bar{\mu}\,\frac{1}{s}), \quad L^{-1}(3\bar{K}\,\frac{1}{s}), \quad L^{-1}(\bar{\lambda}\,\frac{1}{s})$$

are basically relaxation functions.

Let the relaxation functions ϕ, ϕ_1, ϕ_2 be defined as follows:

$$\begin{gathered} \phi(t) = L^{-1}\left(\frac{\bar{Q}}{\bar{P}}\,\frac{1}{s}\right), \quad \phi_1(t) = L^{-1}\left(\frac{\overline{Q'}}{\overline{P'}}\,\frac{1}{s}\right), \\ \phi_2(t) = L^{-1}\left[\frac{1}{3}\left(\frac{\overline{Q'}}{\overline{P'}} - \frac{\bar{Q}}{\bar{P}}\right)\right]. \end{gathered} \tag{84}$$

In turn, the inverted expressions associated with $2\overline{\mu}(s)$, $3\overline{K}(s)$ and $\overline{\lambda}(s)$ are

$$
\begin{aligned}
2\mu &\sim D\delta(t) * \phi(t), \\
3K &\sim D\delta(t) * \phi_1(t), \qquad (85) \\
\lambda &\sim D\delta(t) * \phi_2(t),
\end{aligned}
$$

where

$$\phi_2(t) = \tfrac{1}{3}\left[\phi_1(t) - \phi(t)\right],$$

$*$ is the convolution operator, and the symbol $\sim$ is to be read "corresponds to". The Dirac delta function $\delta(t)$ is defined by

$$\delta(t) = \frac{d\,H(t)}{dt},$$

where

$$
\begin{aligned}
H(t) &= 0, \quad t \leq 0, \\
&= 1, \quad t > 0.
\end{aligned}
$$

Then,

$$L\left[\delta(t)\right] = \int_0^\infty e^{-st}\delta(t)dt = 1.$$

The $L\left[\delta(t)\right]$ can be justified by the use of the Stieltjes integral, but $L\left[\delta'(t)\right]$ requires the use of distribution theory. Thus

$$L\left[\delta^{(k)}(t)\right] = \int_0^\infty e^{-st}\delta^{(k)}(t)dt = (-1)^k \frac{d^k}{dt^k}e^{-st}\Bigg|_{t=0} = s^k$$

where $\delta^{(k)}(t) = \dfrac{d^{k+1}H(t)}{dt^{k+1}}$.

The convolution of $\delta(t)$ and its derivatives with a continuous function $\phi(t)$ is defined as follows:

$$\delta(t) * \phi(t) = \phi(t),$$
$$\delta^{(k)}(t) * \phi(t) = \phi^{(k)}(t) + \sum_{j=0}^{k-1} D^{k-j-1}\delta(t)\phi^{(j)}(0),$$

where

$$\phi^{(0)}(0) = \phi(0),$$

and

$$D^{(0)}\delta(t) = \delta(t).$$

The inverses of the transformed elastic operators can also be given special significance for they can be related to the creep functions ψ, ψ_1, ψ_2 where

$$\psi(t) = L^{-1}\left(\frac{\bar{P}}{\bar{Q}}\frac{1}{s}\right), \qquad \psi_1(t) = L^{-1}\left(\frac{\bar{P'}}{\bar{Q'}}\frac{1}{s}\right),$$

(86)

$$\psi_2(t) = L^{-1}\left[3\,\frac{1}{\left(\frac{\bar{Q'}}{\bar{P'}} - \frac{\bar{Q}}{\bar{P}}\right)}\,\frac{1}{s}\right].$$

Therefore,

$$\frac{1}{2\mu} \sim D\delta(t) * \psi(t)$$

(87)

$$\frac{1}{3k} \sim D\delta(t) * \psi_1(t),$$

$$\frac{1}{\lambda} \sim D\delta(t) * \psi_2(t).$$

The relaxation functions are defined in terms of the generalized Maxwell model, and the creep functions in terms of the generalized Voigt model. Moreover, by virtue of the definitions of $2\bar{\mu}$, $\bar{\lambda}$, $3\bar{K}$ and their reciprocals the two sets of functions (85) and (87) are connected by

$$D^2\delta(t) * \phi(t) * \psi(t) = 1,$$
(88) $$D^2\delta(t) * \phi_1(t) * \psi_1(t) = 1,$$
$$D^2\delta(t) * \phi_2(t) * \psi_2(t) = 1,$$

where

$$D^2\delta(t) = D\delta(t) * D\delta(t),$$

and

$$D^2\delta(t) * \phi(t) * \psi(t) = D^2\delta(t) * G(t),$$
$$G(t) = \phi(t) * \psi(t).$$

Using (85) and (87), other elastic constants can be associated with time operators. The constants $\frac{1}{E}$ and $\frac{\sigma}{E}$ can be associated with creep phenomena since

(89) $$\frac{1}{E} \sim \frac{1}{3}\left\{D\delta(t) * \left[2\,\psi(t) + \psi_1(t)\right]\right\},$$

and

(90) $$\frac{\sigma}{E} \sim \frac{1}{3}\left\{D\delta(t) * \left[\psi(t) - \psi_1(t)\right]\right\}.$$

All other elastic constants, however, are related to mixed behavior, that is, a combination of creep and relaxation involving both shear and dilatation. A few of these are listed below:

$$\frac{1}{1+\sigma} \sim \frac{1}{3}\left[2 + D^2\delta(t) * \psi_1(t) * \phi(t)\right],$$

$$\frac{1}{\sigma} \sim D^2\delta(t) * \phi_1(t) * \psi_2(t),$$

(91) $$\lambda + \mu \sim \frac{1}{6}\,D\delta(t) * \phi_3(t),$$

$$\frac{1}{\lambda+\mu} \sim 6D\delta(t) * \psi_1(t),$$

$$\sigma \sim D^2\delta(t) * \phi_2(t) * \psi_3(t),$$
$$E \sim 3D^3\delta(t) * \phi(t) * \phi_1(t) * \psi_3(t),$$

where

$$\phi_3(t) = 2\phi_1(t) + \phi(t),$$

and

$$D^2\delta(t) * \phi_3(t) * \psi_3(t) = 1.$$

Given a particular material the functions ϕ, ϕ_1, ϕ_2, ϕ_3, ψ, ψ_1, ψ_2, and ψ_3 can be readily obtained. In turn, (85), (87), and (91) can be determined. The above information can then be used to find the viscoelastic solution directly from the modified elastic solution by taking the convolution of the related elastic constants with the boundary conditions whenever convolution is possible.

As an example, consider a beam of length ℓ and constant cross-section stretched by its own weight. Let the stress system be

$$\mathcal{T}_{33} = \rho_0 g a_3, \quad \mathcal{T}_{ij} = 0, \quad ij \neq 33; \tag{92}$$

and the boundary conditions

$$\begin{aligned} &\mathcal{T}_{33} = 0, \quad a_3 = 0; \\ &\mathcal{T}_{33} = \rho_0 g \ell, \quad a_3 = \ell; \\ &\mathcal{T}_{i\alpha} n_\alpha = 0 \text{ on the lateral surface.} \end{aligned} \tag{93}$$

For initial conditions let the deformation be purely elastic at $t = 0$.

In the modified elastic problem,

$$\overline{\mathcal{T}}_{33} = \rho_0 g a_3 \frac{1}{s};$$

hence, the corresponding transformed displacements are

$$(94)\qquad \begin{aligned} \bar{u}_\alpha &= \rho_0 g \left(\frac{\bar{\sigma}}{\bar{E}}\right)\frac{1}{s}\, a_\alpha a_3, \\ \bar{u}_3 &= \frac{1}{2}\,\rho_0 g \left[\frac{1}{\bar{E}}(a_3^2 - \ell^2) + \frac{\bar{\sigma}}{\bar{E}}\, a_\alpha a_\alpha\right]. \end{aligned}$$

All that need be done to invert (94) is to determine $\frac{\sigma}{E} * 1$ and $\frac{1}{E} * 1$. Therefore,

$$(95)\qquad \begin{aligned} u_\alpha &= \frac{1}{3}\,\rho_0 g \left[\Psi(t) - \Psi_1(t)\right] a_\alpha a_3, \\ u_3 &= \frac{1}{6}\,\rho_0 g \left\{\left[2\Psi(t) + \Psi_1(t)\right](a_3^2 - \ell^2) + \right. \\ &\qquad \left.\left[\Psi(t) - \Psi_1(t)\right] a_\alpha a_\alpha\right\}, \end{aligned}$$

since

$$\begin{aligned} D\delta(t) * 1 &= \delta(t), \\ \delta(t) * \Psi(t) &= \Psi(t). \end{aligned}$$

A METHOD FOR OBTAINING SECOND ORDER SOLUTIONS FOR QUASI-STATIC PROBLEMS

Let[1] the displacements v_i and stresses $\mathcal{T}_{ij}$ be the solution to a linear viscoelastic problem in which the acceleration terms are assumed to be negligible. That is, v_i and $\mathcal{T}_{ij}$ satisfy the following conditions within the body:

$$
\begin{aligned}
\epsilon'_{ij} &= \tfrac{1}{2}(v_{i,j} + v_{j,i}), \\
R\,\mathcal{T}_{ij} &= 2M\,\epsilon'_{ij} + N\,\epsilon'_{ss}\delta_{ij}, \\
\mathcal{T}_{ji,j} &+ \rho_0 F_i = 0, \\
\epsilon'_{ij,k\ell} + \epsilon'_{k\ell,ij} &- \epsilon'_{\ell j,ki} - \epsilon'_{ik,j\ell} = 0;
\end{aligned}
\tag{96}
$$

the boundary conditions

$$
\begin{aligned}
T_i &= \mathcal{T}_{ji}n_j \text{ wherever stresses are prescribed,} \\
v_i &= f_i(a_1, a_2, a_3, t) \text{ wherever displacements are prescribed;}
\end{aligned}
\tag{97}
$$

and elastic initial conditions.

The second order stresses $\mathcal{T}'_{ij}$ and displacement gradients α'_{ij} resulting from the first order displacements v_i can be obtained from

$$\alpha'_{ij} = v_{i,s}v_{j,s}, \tag{98}$$

and

---

[1]The basic tenets of this method are due to Rivlin [8].

$$R\,\mathcal{T}'_{ij} = 2M\alpha'_{ij} + N\alpha'_{ss}\delta_{ij}$$

$$(99) \qquad + \left[\,2(M+N)\,\epsilon'_{ij} - N\,\epsilon'_{ss}\delta_{ij}\right]\epsilon'_{ss}$$

$$+ 4M(E'_{ij} - E'_{ss}\delta_{ij}).$$

Initial conditions are again needed to calculate $\mathcal{T}'_{ij}$. They will be chosen so that no initial terms remain in the transform of (99); this does not necessarily imply zero initial conditions.

In order to maintain the displacements v_i on the basis of the second order theory, a set of body forces F'_i and surface tractions T'_i must be applied. These body forces and surface tractions are obtained from (77) by setting

$$\frac{\partial^2 u_i}{\partial t^2} = 0,$$

replacing T_i and F_i by T'_i and F'_i, and using the first order displacements v_i and stresses $_{ij}$. Thus,

$$(100) \qquad \left[(1 + \epsilon'_{ss})\delta_{jk} - v_{j,k}\right]\mathcal{T}_{ki,j} + \mathcal{T}'_{ji,j} + \rho_0 F'_i = 0$$

and

$$(101) \quad T'_i = \left[(1 + \epsilon'_{ss})\delta_{jk} - v_{j,k}\right]\mathcal{T}_{ki,j}n_j + \mathcal{T}'_{ji,j}n_j.$$

The actual body forces F_i and surface tractions T_i were already specified in the formulation of the first order problem; hence, the additional body forces $F'_i - F_i$ and surface tractions $T'_i - T_i$ necessary to maintain the displacements v_i are

$$(102) \quad -\rho_0(F_i' - F_i) = (\epsilon_{ss}'\delta_{kj} - v_{j,k})\mathcal{T}_{ki,j} + \mathcal{T}_{ji,j}',$$

and

$$(103) \quad T_i' - T_i = (\epsilon_{ss}'\delta_{kj} - v_{j,k})\mathcal{T}_{ki}n_j + \mathcal{T}_{ji}'n_j.$$

Since these forces are not present, a new problem is formulated to determine the displacements w_i which the reverse of the body forces and surface tractions would produce according to linear viscoelastic theory.

The second problem consists of solving the set of equations

$$(104) \quad \begin{aligned} \epsilon_{ij}'' &= \tfrac{1}{2}(w_{i,j} + w_{j,i}), \\ R\,\mathcal{T}_{ij}'' &= 2M\,\epsilon_{ij}'' + N\,\epsilon_{ss}''\delta_{ij}, \\ \mathcal{T}_{ji,j}'' &+ \rho_0(F_i - F_i') = 0, \\ \epsilon_{ij,k\ell}'' + \epsilon_{k\ell,ij}'' &- \epsilon_{\ell j,ki}'' - \epsilon_{ik,j\ell}'' = 0 \end{aligned}$$

in the body; the stress boundary conditions

$$(105) \quad T_i - T_i' = \mathcal{T}_{ji}''n_j$$

on the surface; and the usual elastic initial conditions.

There is one obvious difficulty inherent in the formulation of the second linear problem, since both the body forces and surface tractions are specified without assurance that they are compatible. In short, it may be impossible to achieve equilibrium with the specified conditions. Consequently, additional surface tractions may have to be specified in order to balance the body forces.

EXAMPLES

The first two examples will consider a viscoelastic material which in the linear treatment can be represented by a four-parameter model in shear, and a purely elastic model under dilatation, i.e.,

$$P' = 1, \quad P = D^2 + AD + B, \qquad (106)$$
$$Q' = H, \quad Q = CD^2 + GD, \quad D = \frac{\partial}{\partial t},$$

where

A, B, C, G, and H are all constants.

A, B, C, and G can be interpreted either in terms of the parameters of the double Maxwell model, or in terms of the generalized Voigt model. In the second order problem, the operators P, P', Q', Q lose the simple physical significance attributed to them in the linear case; instead they become merely a set of operators which connect the stress, strain, and time. Because of the way in which the second order problem has been constructed, non-linearity does not occur in the time operators directly.

The operators R, M, and N which appear in the stress-strain relations are

$$R = PP' = D^2 + AD + B,$$
$$M = \frac{1}{2}P'Q = \frac{1}{2}(CD^2 + GD), \qquad (107)$$
$$N = \frac{1}{3}(PQ' - QP') = \frac{1}{3}\left[H(D^2 + AD + B) - (CD^2 + GD)\right].$$

As the first example, consider the simple shear of a cube in a plane parallel to one face for which displacement boundary conditions are prescribed.

$$
\begin{aligned}
&u_2(0,a_2,a_3,t)=u_2(\ell,a_2,a_3,t)=u_2(a_1,0,a_3,t)=u_2(a_1,\ell,a_3,t)=0,\\
&u_2(a_1,a_2,0,t)=u_2(a_1,a_2,\ell,t)=u_3(0,a_2,a_3,t)=u_3(\ell,a_2,a_3,t)=0,\\
(108)\quad &u_3(a_1,0,a_3,t)=u_3(a_1,\ell,a_3,t)=u_3(a_1,a_2,0,t)=u_3(a_1,a_2,\ell,t)=0,\\
&u_1(0,a_2,a_3,t)=u_1(\ell,a_2,a_3,t)=u_1(a_1,a_2,0,t)=u_1(a_1,a_2,\ell,t)=Ka_2,\\
&u_1(a_1,\ 0,\ a_3,\ t) = 0, \quad u_1(a_1,\ \ell\ ,\ a_3,\ t) = K\ \ ,
\end{aligned}
$$

where

$$K = \text{constant},$$

$$\ell = \text{length of side of the cube.}$$

A compatible set of displacements satisfying displacement boundary conditions are

$$(109)\qquad u_1 = Ka_2, \quad u_2 = u_3 = 0.$$

This displacement system produces strains

$$
\epsilon_{ij} = \frac{1}{2}\begin{bmatrix} 0 & K & 0 \\ K & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \alpha_{ij} = \frac{1}{2}\begin{bmatrix} K^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
$$

$$
(110)\qquad E_{ij} = \frac{1}{4}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -K^2 \end{bmatrix},
$$

$$\epsilon_{ss} = 0, \quad \alpha_{ss} = \tfrac{1}{2}K^2, \quad E_{ss} = -\ \tfrac{1}{4}K^2;$$

which in turn produce a set of stresses satisfying equilibrium:

$$
\begin{aligned}
R\sigma_{11} &= \tfrac{1}{2}(N + 4M)K^2, \\
R\sigma_{22} &= \tfrac{1}{2}(N + 2M)K^2, \\
R\sigma_{33} &= \tfrac{1}{2} NK^2, \\
R\sigma_{12} &= MK, \\
R\sigma_{23} &= R\sigma_{31} = 0.
\end{aligned} \tag{111}
$$

The time dependent stresses can be obtained by transforming (111) and then inverting. Since this is a relaxation problem, the double Maxwell model clearly depicts the behavior. The solution is

$$
\begin{aligned}
\sigma_{11} &= \tfrac{1}{6}\left[H + 5\phi(t) \right] K^2, \\
\sigma_{22} &= \tfrac{1}{6}\left[H + 2\phi(t) \right] K^2, \\
\sigma_{33} &= \tfrac{1}{6}\left[H - \phi(t) \right] K^2, \\
\sigma_{12} &= \tfrac{1}{2}\phi(t)K, \\
\sigma_{23} &= \sigma_{31} = 0,
\end{aligned} \tag{112}
$$

where $\phi(t)$ is the recollection function for relaxation (26).

When the stresses are prescribed on the surface of a cube, the ensuing problem is that of creep. It is precisely this type of problem which motivates the formulation of a second order theory. The linear problem associated with a single prescribed set of surface shears consists of finding the displacements v_i and stresses $\mathcal{T}_{ij}$ which result from the stress boundary conditions

$$T_1 = T_2 = T_3 = 0, \quad a_3 = 0, \text{ and } a_3 = \ell \,;$$

(113)
$$T_1 = T_3 = 0, \quad T_2 = \pm p, \quad a_1 = 0, \text{ and } a_1 = \ell \,;$$

$$T_2 = T_3 = 0, \quad T_1 = \pm p, \quad a_2 = 0, \text{ and } a_2 = \ell \,.$$

A stress field which satisfies the above stress boundary conditions and equilibrium is

(114)
$$\mathcal{T}_{12} = p, \qquad \mathcal{T}_{ij} = 0, \quad ij \neq 12,\ 21.$$

The transform of (114) looks just like the corresponding elastic problem; hence the solution (excluding rigid body motions) for the transformed displacements is

(115)
$$\bar{v}_1 = p\,\frac{1}{\bar{\mu}s}\,a_2,$$
$$\bar{v}_2 = \bar{v}_3 = 0.$$

Inverting (115),

(116)
$$v_1 = 2p\,\psi(t)a_2,$$
$$v_2 = v_3 = 0,$$

where $\psi(t)$ is the memory function for creep (24).

If K in (109) is replaced by $2p\,\psi(t)$, (116) is identical with (109). Consequently the resulting second order stresses are

(117)
$$R\,\mathcal{T}'_{11} = 2p^2(N + 4M)\,\psi^2,$$
$$R\,\mathcal{T}'_{22} = 2p^2(N + 2M)\,\psi^2,$$
$$R\,\mathcal{T}'_{33} = 2p^2N\,\psi^2,$$
$$R\,\mathcal{T}'_{12} = R\,\mathcal{T}'_{23} = R\,\mathcal{T}'_{13} = 0.$$

Transforming (117) while taking into account (112),

$$(118)\quad \begin{aligned} \overline{\tau'_{11}} &= \tfrac{2}{3}\, p^2 (H + 5s\bar{\phi})\, \overline{\psi^2}, \\ \overline{\tau'_{22}} &= \tfrac{2}{3}\, p^2 (H + 2s\bar{\phi})\, \overline{\psi^2}, \\ \overline{\tau'_{33}} &= \tfrac{2}{3}\, p^2 (H - s\bar{\phi})\, \overline{\psi^2}, \\ \overline{\tau'_{12}} &= \overline{\tau'_{23}} = \overline{\tau'_{31}} = 0, \end{aligned}$$

where $\bar{\phi}(s)$ is given by (25) and $\overline{\psi}(s)$ by (23).

The time dependent stresses determined by (118) are

$$(119)\quad \begin{aligned} \tau'_{11} &= \tfrac{2}{3}\, p^2 \left[H\psi^2(t) + 5F(t) \right], \\ \tau'_{22} &= \tfrac{2}{3}\, p^2 \left[H\psi^2(t) + 2F(t) \right], \\ \tau'_{33} &= \tfrac{2}{3}\, p^2 \left[H\psi^2(t) - F(t) \right], \\ \tau'_{12} &= \tau'_{23} = \tau'_{31} = 0, \end{aligned}$$

where

$$(120)\quad \begin{aligned} \psi^2(t) = {} & \left(\frac{1}{E_1} + \frac{1}{E_2}\right)^2 + \frac{2}{\eta_1}\left(\frac{1}{E_1} + \frac{1}{E_2}\right)t + \frac{1}{\eta_1^2}\, t^2 \\ & - \frac{2}{E_2}\left(\frac{1}{E_1} + \frac{1}{E_2}\right) e^{-\frac{E_2}{\eta_2}t} - \frac{2}{\eta_1 E_2}\, t e^{-\frac{E_2}{\eta_2}t} \\ & + \frac{1}{E_2^2}\, e^{-2\frac{E_2}{\eta_2}t}, \end{aligned}$$

and

$$F(t) = \left(\frac{1}{E_1} + \frac{1}{E_2}\right) + \frac{2}{\eta_1}t - \frac{1}{E_1 E_2}\left(E'_1 e^{-\frac{E'_1}{\eta'_1}t} + E'_2 e^{-\frac{E'_2}{\eta'_2}t}\right)$$

$$- \frac{1}{\eta_1}\left(\frac{1}{E_1} + \frac{1}{E_2}\right) \left[\eta_1'\left(1 - e^{-\frac{E_1'}{\eta_1'}t}\right) + \eta_2'\left(1 - e^{-\frac{E_2'}{\eta_2'}t}\right)\right]$$

$$- \frac{2}{\eta_1\eta_2} \frac{E_1'}{\left(\frac{E_2}{\eta_2} - \frac{E_1'}{\eta_1'}\right)^2} \left\{\left[\left(\frac{E_2}{\eta_2} - \frac{E_1}{\eta_1}\right)t-1\right] e^{-\frac{E_2}{\eta_2}t} + e^{-\frac{E_1'}{\eta_1'}t}\right\}$$

$$- \frac{2}{\eta_1\eta_2} \frac{E_2'}{\left(\frac{E_2}{\eta_2} - \frac{E_2'}{\eta_2'}\right)^2} \left\{\left[\left(\frac{E_2}{\eta_2} - \frac{E_2'}{\eta_2'}\right)t-1\right] e^{-\frac{E_2}{\eta_2}t} + e^{-\frac{E_2'}{\eta_2'}t}\right\}$$

(121)

$$- \frac{1}{\eta_2}\left(\frac{1}{E_1} + \frac{1}{E_2}\right) \left[\frac{E_1'}{\left(\frac{E_2}{\eta_2} - \frac{E_1'}{\eta_1'}\right)}\left(e^{-\frac{E_2}{\eta_2}t} - e^{-\frac{E_1'}{\eta_1'}t}\right)\right.$$

$$\left. + \frac{E_2'}{\left(\frac{E_2}{\eta_2} - \frac{E_2'}{\eta_2'}\right)}\left(e^{-\frac{E_2}{\eta_2}t} - e^{-\frac{E_2'}{\eta_2'}t}\right)\right]$$

$$+ \frac{2}{E_2\eta_2} \left[\frac{E_1'}{\left(2\frac{E_2}{\eta_2} - \frac{E_1'}{\eta_1'}\right)}\left(e^{-2\frac{E_2}{\eta_2}t} - e^{-\frac{E_1'}{\eta_1'}t}\right)\right.$$

$$\left. + \frac{E_2'}{\left(2\frac{E_2}{\eta_2} - \frac{E_2'}{\eta_2'}\right)}\left(e^{-2\frac{E_2}{\eta_2}t} - e^{-\frac{E_2'}{\eta_2'}t}\right)\right] .$$

The behavior of the second order stresses at $t = 0$ and as $t \rightarrow \infty$ can be ascertained from $\psi^2(t)$ and $F(t)$.

$$\psi^2(0) = \frac{1}{E_1^2}, \quad F(0) = \frac{1}{E_1}, \tag{122}$$

and

$$\begin{aligned} &\lim_{t\to\infty} \psi^2(t) = \lim_{t\to\infty} \left[\left(\frac{1}{E_1} + \frac{1}{E_2}\right)^2 + \frac{2}{\eta_1}\left(\frac{1}{E_1} + \frac{1}{E_2}\right)t + \frac{1}{\eta_1^2}t^2 \right], \\ &\lim_{t\to\infty} F(t) = \lim_{t\to\infty} \frac{2}{\eta_1}t \end{aligned} \tag{123}$$

Thus, $F(t)$ governs the stress when $t \sim 0$, but as t increases $\psi^2(t)$ begins to dominate.

From (114), (116), and (119) the additional body forces given by (102) are

$$-\rho_0(F_i' - F_i) = 0, \quad i = 1, 2, 3; \tag{124}$$

and the additional surface tractions required by (103) are

$$\begin{aligned} &T_1' - T_1 = \pm\, 2p^2\left[\tfrac{1}{3}(H\psi^2 + 5F) - \psi\right], \\ &T_2' - T_2 = T_3' - T_3 = 0, \text{ when } a_1 = \ell, \text{ and } a_1 = 0 \text{ respectively;} \\ &T_2' - T_2 = \pm\, \tau_{22}', \\ &T_1' - T_1 = T_3' - T_3 = 0, \text{ when } a_2 = \ell, \text{ and } a_2 = 0 \text{ respectively;} \\ &T_3' - T_3 = \pm\, \tau_{33}', \\ &T_1' - T_1 = T_2' - T_2 = 0, \text{ when } a_3 = \ell, \text{ and } a_3 = 0 \text{ respectively.} \end{aligned} \tag{125}$$

The second order problem is as follows: Find the stresses τ_{ij}'' and displacements w_i when the body forces $\rho_0(F_i - F_i') = 0$, and the cube is subjected to three uniaxial compressive stresses given by

$$
(126) \qquad
\begin{aligned}
\mathcal{T}''_{11} &= T_1 - T'_1 \text{ along the } A_1 \text{ axis,} \\
\mathcal{T}''_{22} &= T_2 - T'_2 \quad '' \quad '' \quad A_2 \quad '' \quad , \\
\mathcal{T}''_{33} &= T_3 - T'_3 \quad '' \quad '' \quad A_3 \quad '' \quad .
\end{aligned}
$$

This problem can be solved by superimposing the solutions for each of the three compressive stresses acting individually. Consider first the problem of a uniaxial stress along the A_1 axis. Transforming the first of equations (125) with due regard for (126),

$$
(127) \qquad
\begin{aligned}
\overline{\mathcal{T}''_{11}} &= -2p^2 \left[\tfrac{1}{3}(H\overline{\Psi^2} + 5\overline{F}) - \overline{\Psi} \right] ; \\
\overline{\mathcal{T}''_{ij}} &= 0, \quad ij \neq 11.
\end{aligned}
$$

The solution to the equivalent elastic problem is

$$
(128) \qquad
\begin{aligned}
\overline{w_2} = \overline{w_3} &= -\frac{\overline{\sigma}}{\overline{E}} \overline{\mathcal{T}''_{11}} a_\alpha, \\
\overline{w_1} &= \frac{1}{\overline{E}} \overline{\mathcal{T}''_{33}} a_3
\end{aligned}
$$

Inverting (128),

$$
(129) \qquad
\begin{aligned}
w_1 &= -\frac{2}{3} p^2 \left[\frac{2}{3} \xi(t) + \frac{10}{3} \zeta(t) - 2\gamma(t) \right] + \frac{1}{3H} \,{}''_{11}, \\
w_2 = w_3 &= \frac{2}{3} p^2 \left[\frac{1}{3} \xi(t) + \frac{5}{3} \zeta(t) - \gamma(t) \right] + \frac{1}{3H} \mathcal{T}''_{11}.
\end{aligned}
$$

By a similar process the displacements due to the stresses along A_2 and A_3 can be calculated. Adding the three solutions,

$$
(130) \qquad
\begin{aligned}
w_1 &= -\frac{2}{3} p^2 \left[3\,\zeta(t) + 2\gamma(t) \right] + \frac{1}{3H} \mathcal{T}''_{ii}, \\
w_2 &= -\frac{2}{3} p^2 \gamma(t) + \frac{1}{3H} \mathcal{T}''_{ii},
\end{aligned}
$$

$$w_3 = -\frac{2}{3} p^2 \left[-3\,\varsigma(t) + \Upsilon(t) \right] + \frac{1}{H}\mathcal{T}''_{ii},$$

where

$$\mathcal{T}''_{ii} = -2p^2(H\psi^2 + 2F - \psi),$$
$$\xi(t) = \psi^2 * \psi \quad ,$$
(131) $$\varsigma(t) = \psi * F,$$
$$\Upsilon(t) = \psi * \psi \quad .$$

The second order solution for the displacements is

$$u_1 = v_1 + w_1,$$
(132) $$u_2 = w_2,$$
$$u_3 = w_3.$$

The displacements (132) represent a valid solution only when the acceleration terms are negligible. Considering the acceleration as a special type of second order effect, the only non-vanishing acceleration term resulting from (116) is

(133) $$\frac{\partial^2 v_1}{\partial t^2} = 2\,p\,\mathcal{A}(t)a_2,$$

where

$$\mathcal{A}(t) = -\frac{E_2}{\eta_2^2}\, e^{-\frac{E_2}{\eta_2}t}.$$

The acceleration term (133) is a consequence of the Voigt mechanism; it attains its maximum absolute value at $t = 0$, and as t increases the error introduced by neglecting (133) diminishes. Since for $t \sim 0$ the neglected acceleration terms may be of the same order of magnitude as the second order effects previously obtained, the validity of the latter is considerably in doubt.

An estimate of the contribution of the acceleration term can be obtained by formulating the following problem: Let $\mathcal{T}_{ij}$ be the stresses and v_i the displacements associated with the linear problem. Let the acceleration resulting from the linear displacements v_i be represented by a set of body forces

$'F_i = -\dfrac{\partial^2 v_i}{\partial t^2}$, and let $'\mathcal{T}_{ij}$ be the stresses associated with

the new stress boundary value problem

$$'\mathcal{T}_{ji,j} + \rho_0 {'F_i} = 0, \tag{134}$$

$$'T_i = {'\mathcal{T}_{ji}} n_j. \tag{135}$$

The stresses $'\mathcal{T}_{ij}$ are a consequence of the neglected acceleration, and do in fact exist in the body (though $'\mathcal{T}_{ij}$ are only approximations). The stress boundary conditions $'T_i$ are a set of surface tractions which must be applied during deformation in order to preserve equilibrium. Rather than specifying the $'T_i$ beforehand, they will be obtained from a set of stresses satisfying (134) and the Beltrami-Michell equations. Although many possible solutions to an elastic problem exist when the boundary conditions are not specified, the solution is unique when they are specified. Equilibrium can be maintained by any number of different types of surface tractions, each one of which gives rise to a different set of stresses within the body, but which all satisfy the same basic equations.

Returning to the problem of a cube subjected to a constant shear, the body forces $'F_i$ are

$$'F_1 = Ka_2, \quad 'F_2 = 'F_3 = 0, \tag{136}$$

where

$$K = 2p\,\Omega(t).$$

The related modified elastic problem consists of the equilibrium equations

$$\begin{aligned} '\overline{\mathcal{T}}_{ji,j} &= \rho_0 \overline{K} a_2 \\ '\overline{\mathcal{T}}_{j2,j} &= '\overline{\mathcal{T}}_{j3,j} = 0; \end{aligned} \tag{137}$$

the Beltrami-Michell equations

$$\begin{aligned} '\overline{\mathcal{T}}_{ij} + \frac{1}{1+\overline{\sigma}}\, '\overline{\mathcal{T}}_{ss,ij} &= 0, \\ '\overline{\mathcal{T}}_{12,ss} + \frac{1}{1+\overline{\sigma}}\, '\overline{\mathcal{T}}_{ss,12} &= \rho_0 \overline{K}; \end{aligned} \tag{138}$$

and the unspecified boundary conditions.

A system of stresses satisfying (137) and (138) is

$$\begin{aligned} '\overline{\mathcal{T}}_{12} &= \tfrac{1}{2}\rho_0 \overline{K} a_2^2, \\ '\overline{\mathcal{T}}_{11} = '\overline{\mathcal{T}}_{22} &= '\overline{\mathcal{T}}_{33} = '\overline{\mathcal{T}}_{23} = '\overline{\mathcal{T}}_{31} = 0. \end{aligned} \tag{139}$$

The corresponding surface tractions are

$$'\overline{T}_1 = '\overline{\mathcal{T}}_{21} n_2, \quad '\overline{T}_2 = \overline{\mathcal{T}}_{12} n_1, \quad '\overline{T}_3 = 0.$$

In turn, the set of stresses (139) produce displacements

(140)
$$'\bar{v}_1 = \frac{1}{3}\rho_0 \frac{1+\bar{\sigma}}{\bar{E}}\bar{K}a_2^3,$$
$$'\bar{v}_2 = {}'\bar{v}_3 = 0,$$

where

$$\bar{K} = -2p\frac{E_2}{\eta_2^2}\frac{1}{(s+\frac{E_2}{\eta_2})}.$$

Therefore,

(141)
$$'v_1 = -2\rho_0 p\frac{E_2}{{}_2^{\,2}}\Gamma(t)a_2^3,$$
$$'v_2 = {}'v_3 = 0,$$

where

$$\Gamma(t) = \left(\frac{1}{E_1}+\frac{2}{\eta_1 E_2}+\frac{1}{\eta_2}t\right)e^{-\frac{E_2}{\eta_2}t} - \frac{\eta_2}{\eta_1 E_2},$$

and

(142)
$$\Gamma(0) = \frac{1}{E_1}, \quad \lim_{t\to\infty}\Gamma(t) = -\frac{\eta_2}{\eta_1 E_2}.$$

The necessary surface tractions are

(143)
$$'T_1 = 0, \qquad a_2 = 0;$$
$$'T_2 = {}'T_3 = 0,$$
$$'T_1 = \rho_0 p\,\Omega(t)\,\ell^2, \qquad a_2 = \ell;$$
$$'T_2 = {}'T_3 = 0,$$
$$'T_2 = \pm\rho_0 p\,\Omega(t)a_2^2, \quad a_1 = \ell, \quad a_1 = 0 \text{ respectively;}$$
$$'T_1 = {}'T_3 = 0,$$

$$'T_1 = 'T_2 = 'T_3 = 0, \quad a_3 = 0, \quad \dot{a}_3 = \ell .$$

Equations (141) and (142) show that a permanent deformation results even though the acceleration decays exponentially to zero.

Comparing equations (130) and (141), no essential difference can be discerned in the time solutions, but a difference in the power of the spatial components as well as in the shear stress p is noticeable. The square of the applied linear stress occurs in (130), while only a linear stress term appears in (141). Also no spatial components appear in (130), but they are of the third degree in (141). Hence, when $t \sim 0$ the second order solution (130) gives reasonable results for a Voigt material when the shear stress is large and the sample size small; but if the shear stress is small and the sample size large, the Voigt behavior may as well be excluded.

In conclusion it appears that the second order correction is best suited for simple Maxwell behavior unless experimental design has minimized the influence of acceleration.

The last example is concerned with a viscoelastic material which can be represented by a Maxwell model in shear, and a purely elastic model under dilatation, i.e.,

$$\begin{aligned} &P' = 1, && P = D + \frac{E'}{\eta'}, && \\ (144) \qquad &Q' = H, && Q = E'D, && D = \frac{\partial}{\partial t}, \\ &H = \text{constant}, && && \end{aligned}$$

and

$$R = D + \frac{E'}{\eta^{\prime}}, \quad M = \frac{1}{2} E'D,$$

$$N = \frac{1}{3}\left[H(D + \frac{E'}{\eta^{\prime}}) - E'D\right].$$

Consider a circular beam of length ℓ and radius a, subjected to a constant uniaxial stress. On the basis of the linear theory the required stress system is

(145) $$\tau_{33} = p, \quad \tau_{ij} = 0, \quad ij \neq 33,$$

and the corresponding displacements

(146) $$v_\alpha = -p\,\frac{\sigma}{E}\,a_\alpha, \qquad v_3 = p\,\frac{1}{E}\,a_3.$$

By (89), (90) and the fact that the stress is constant, the displacements in the viscoelastic problem are

(147) $$v_\alpha = -\,\zeta(t)\,a_\alpha, \qquad v_3 = \beta(t)a_3,$$

where

$$\zeta(t) = p\left[L^{-1}\left(\frac{\bar{\bar{\sigma}}}{\bar{\bar{E}}}\right) * 1\right] = \frac{1}{3}\,p\left[\psi(t) - \frac{1}{H}\right],$$

$$\beta(t) = p\left[L^{-1}\left(\frac{1}{\bar{\bar{E}}}\right) * 1\right] = \frac{1}{3}\,p\left[2\,\psi(t) + \frac{1}{H}\right],$$

$$\psi(t) = \frac{1}{E} + \frac{1}{\eta}t.$$

Substituting the first order displacements into the expression for the strains,

$$\epsilon'_{ij} = \begin{bmatrix} -\zeta & 0 & 0 \\ 0 & -\zeta & 0 \\ 0 & 0 & \beta \end{bmatrix}, \quad \alpha'_{ij} = \frac{1}{2}\begin{bmatrix} \zeta^2 & 0 & 0 \\ 0 & \zeta^2 & 0 \\ 0 & 0 & \beta^2 \end{bmatrix},$$

$$E'_{ij} = \frac{1}{4}\begin{bmatrix} -\beta & 0 & 0 \\ 0 & -\beta & 0 \\ 0 & 0 & \zeta^2 \end{bmatrix}, \tag{148}$$

$$\epsilon'_{ss} = \beta - 2\zeta, \quad \alpha'_{ss} = \frac{1}{2}(2\zeta^2 + \beta^2),$$

$$E'_{ss} = \frac{1}{4}(\zeta^2 - 2\zeta\beta).$$

Equations (99) and (148) furnish the strains $\mathcal{T}'_{ij}$

$$R\mathcal{T}'_{(\alpha\alpha)} = M(4\zeta^2 - \beta\zeta) + \frac{1}{2}N(2\zeta^2 + 4\beta\zeta - \beta^2),$$

$$R\mathcal{T}'_{33} = M(3\beta^2 - 2\beta\zeta) + \frac{3}{2}N(\beta^2 - 2\zeta^2), \tag{149}$$

$$R\mathcal{T}'_{ij} = 0, \; i \neq j,$$

where the repeated subscripts in parenthesis are not summed.

Transforming (149) and then inverting,

$$\mathcal{T}'_{(\alpha\alpha)} = \frac{1}{9}Hp^2(2\psi^2 - \frac{4}{H}\psi - \frac{1}{H^2}) + \frac{1}{18}p^2(\frac{1}{H} + \frac{7}{H^2}\phi - 2\theta),$$

$$\mathcal{T}'_{33} = \frac{1}{18}Hp^2(2\psi^2 + \frac{8}{H}\psi - \frac{1}{H^2}) + \frac{1}{6}p^2(-\frac{4}{H} + \frac{2}{H^2}\phi - \theta), \tag{150}$$

$$\mathcal{T}'_{ij} = 0, \; i \neq j,$$

where

$$\phi(t) = Ee^{-\frac{E}{\eta}t},$$

$$\theta(t) = \frac{2}{\eta}t + \frac{1}{E}e^{-\frac{E}{\eta}t}.$$

As in the previous problem the necessary additional body forces are zero.

$$-(F'_i - F_i) = 0.$$

The required additional surface tractions according to (111) are

$$(151) \qquad \begin{aligned} &T'_\alpha - T_\alpha = \mathcal{T}'_{(\alpha\alpha)} n_\alpha, \quad n_\alpha = \frac{x_\alpha}{a}, \\ &T'_3 - T_3 = 0 \end{aligned}$$

on the lateral surface of the cylinder, and

$$(152) \qquad \begin{aligned} &T'_\alpha - T_\alpha = 0, \\ &T'_3 - T_3 = (-2\zeta p + \mathcal{T}'_{33}) n_3 \end{aligned}$$

on the plane ends of the cylinder. The system of tractions (152) is equivalent to a radial surface traction Z acting in an outward direction on the lateral surface of the cylinder,

$$Z = \mathcal{T}'_{(\alpha\alpha)},$$

and a normal surface traction along the axis of the cylinder.

The second order problem can now be formulated: Find the displacements w_i and stresses $\mathcal{T}''_{ij}$ when the cylinder is subjected to the stress boundary conditions

$$(153) \qquad Z = - \mathcal{T}'_{(\alpha\alpha)}$$

on the lateral surface of the cylinder, and the tensile stress

$$(154) \qquad T_3 - T'_3 = 2\zeta p - \mathcal{T}'_{33}$$

along the axis of the cylinder.

First consider the problem of a circular cylinder of length ℓ and radius a, under the influence of the radial compression (153) when the cylinder is kept at constant length. Let w denote the radial displacement, and $\mathcal{T}''_{rr}$, $\mathcal{T}''_{r\theta}$, $\mathcal{T}''_{\theta\theta}$ the polar components of the stress tensor.

According to classical elastic theory the solution is

$$
(155)\qquad
\begin{aligned}
{}_1\mathcal{T}''_{rr} &= {}_1\mathcal{T}''_{\theta\theta} = -\ \mathcal{T}'_{11},\\
{}_1\mathcal{T}''_{33} &= -\frac{\lambda}{\mu+\lambda}\ \mathcal{T}'_{11},\\
{}_1\mathcal{T}''_{r\theta} &= {}_1\mathcal{T}''_{\theta z} = {}_1\mathcal{T}''_{rz} = 0,\\
-\ w_r &= \frac{1}{2}\,\frac{1}{\mu+\lambda}\ \mathcal{T}'_{11}r,\\
w_\theta &= w_z = 0;
\end{aligned}
$$

or in rectangular coordinates,

$$
(156)\qquad
\begin{aligned}
{}_1\mathcal{T}''_{(\alpha\alpha)} &= -\ \mathcal{T}'_{11},\\
{}_1\mathcal{T}''_{12} &= {}_1\mathcal{T}''_{23} = {}_1\mathcal{T}''_{31} = 0,\\
{}_1\mathcal{T}''_{33} &= -\frac{\lambda}{\mu+\lambda}\ {}_1\mathcal{T}''_{11},\\
w_\alpha &= -\frac{1}{2}\,\frac{1}{\mu+\lambda}\ \mathcal{T}'_{11}a_\alpha,\\
w_3 &= 0.
\end{aligned}
$$

The relevant terms in the corresponding viscoelastic problem are

$$
(157)\qquad
\begin{aligned}
\overline{{}_1\mathcal{T}''_{33}} &= -\frac{\bar\lambda}{\bar\mu+\bar\lambda}\ \overline{\mathcal{T}'_{11}},\\
\overline{w_\alpha} &= -\frac{1}{2}\,\frac{1}{\bar\mu+\bar\lambda}\ \overline{\mathcal{T}'_{11}}a_\alpha.
\end{aligned}
$$

Hence,

$$
(158)\qquad
\begin{aligned}
w_\alpha &= -\frac{1}{2}\,S(t)a_\alpha,\\
{}_1\mathcal{T}''_{33} &= -\frac{1}{3}(H-E)S(t) + \frac{1}{3}\,\frac{E^2}{\eta}\,e^{-\frac{E}{\eta}t} * \ S(t),
\end{aligned}
$$

where

$$S(t) = \frac{1}{9}p^2\Big[\, 2\,\psi^2 - \frac{7}{H}\psi + \frac{5}{H^2} - \frac{6H}{2H - E}(\psi^2 + \frac{2}{H^2}) + \frac{6E^2H}{\eta(2H - E)}(\psi^2 + \frac{2}{H^2}) \ast e^{-\frac{2HE}{\eta(2H-E)}}\Big].$$

Since ${}_1\tau''_{33}$ is not included in the applied surface tractions, the reversed stresses ${}_1\tau''_{33}$ must be included in the uniaxial stress along A_3. Thus, the final problem is to determine the displacements w_i due to the axial stress

$$\tau''_{33} = 2\zeta p - \tau'_{33} - {}_1\tau''_{33}. \tag{159}$$

The modified elastic solution is

$$\begin{aligned} \overline{w_\alpha} &= -\frac{\overline{\sigma}}{\overline{E}}\,\overline{\tau''_{33}}a_\alpha, \\ \overline{w_3} &= \frac{1}{\overline{E}}\,\overline{\tau''_{33}}a_3 \end{aligned} \tag{160}$$

Hence,

$$\begin{aligned} w_\alpha &= -\zeta_1(t) \ast \tau''_{33}a_\alpha = -S_1(t)a_\alpha, \\ w_3 &= \beta_1(t) \ast \tau''_{33}a_3 = S_2(t)a_3; \end{aligned} \tag{161}$$

where

$$\zeta_1(t) = \frac{1}{3}\Big[(\frac{1}{E} - \frac{1}{H})\delta(t) + \frac{1}{\eta}\Big],$$

$$\beta_1(t) = \frac{1}{3}\Big[(\frac{2}{E} + \frac{1}{H})\delta(t) + \frac{2}{\eta}\Big];$$

and

$$\tau''_{33} = \frac{1}{18}p^2\Big[-2H\,\psi^2 + 4\psi + \frac{2}{\eta}t + (\frac{1}{E^2} - \frac{2}{H^2})\phi\Big] + \frac{1}{3}\Big[(H - E)S(t) + \frac{E^2}{\eta}e^{-\frac{E}{\eta}t} \ast S(t)\Big].$$

The total displacement is

$$(162) \qquad \begin{aligned} u_\alpha &= - \left[\zeta(t) + \tfrac{1}{2} S(t) + S_1(t) \right] a_\alpha, \\ u_3 &= \left[\beta(t) + S_2(t) \right] a_3. \end{aligned}$$

The solution for the displacements w_i contains a polynomial of degree three in t in addition to the exponential decay terms. Hence, for t sufficiently large, the second order solution will exceed in magnitude the first order solution. The second order solution need not be included in the total displacements as long as $v_i > 0.1w_i$, but certainly beyond this point the second order correction is needed; when, however, $w_i > v_i$ the validity of the solution u_i is clearly in doubt.

SUMMARY

The first section of this paper reviews the description of viscoelastic behavior with a special interest in the relationship between the initial conditions and the Laplace transform of the differential expressions. It is shown that initial elastic conditions (terms required by the mechanical models) drop out of the transformed problem, simplifying the resulting inversion.

Integral expressions (31) and (32) are obtained from model studies, but the memory function ψ and recollection function $\emptyset$ can be considered as general descriptions of viscoelastic behavior without regard to model representation; thus, (31) and (32) have a broader interpretation than (20). In fact, the derivations of (39) and (40) given by Leaderman [10] do not depend on model representation, but on the existence of a time dependent elastic history.

After reviewing the formulation of three-dimensional stress-strain laws for linear, homogeneous, isotropic, viscoelastic materials; and introducing the non-linear elastic stress-strain law derived by Rivlin [13], a non-linear viscoelastic stress-strain law containing four basic operators P, Q, P', Q' is proposed. This stress-strain law is based on Rivlin's non-linear elastic stress-strain law containing two elastic constants. It is also assumed that the initial terms

are such that they cancel in the transformed (Laplace) problem.

A method is then developed for obtaining second order corrections to quasi-static viscoelastic problems. Several examples are then presented using this method. From these examples the following conclusions can be drawn: When displacements are prescribed, no inertial terms exist; hence, both the first order and second order solutions are valid for all materials. When the stresses are prescribed, and the inertial terms neglected in the equilibrium equations, the validity of the solution depends on the magnitude of the stress, the size of the sample, and the type of the material. For Maxwell material the neglected inertial term has no effect on the solution. For a Voigt material, however, the second order correction could be of the same order of magnitude as the correction for the neglected acceleration term, depending on the body shape and the boundary conditions.

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